Starter

P(2, −1) and Q(8, 3) are opposite ends of a diameter of a circle. The centre and radius of it are

	Centre	Radius
A.	(5, 2)	$\sqrt{26}$
B.	(5, 1)	$\sqrt{26}$
C.	(5, 1)	$\sqrt{13}$
D.	(5, 2)	$\sqrt{13}$

midpt

$\left(\frac{2+8}{2}, \frac{-1+3}{2}\right) = (5,1)$

centre $(5,1)$

diameter $= \sqrt{(8-2)^2+(3-1)^2} = \sqrt{6^2+4^2} = \sqrt{52}$

$= 2\sqrt{13}$

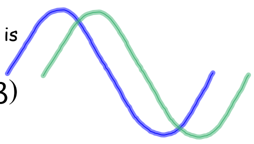
radius $= \sqrt{13}$

Chapter 11: Addition Formulae

Compound Angles

Today's Learning:
By the end of this lesson, you should be able to determine what a compound angle is.

When the graph of $\sin\alpha$ is moved left or right it becomes $y = \sin(\alpha + \beta)$



Angles such as $(\alpha + \beta)$ are called **compound angles**.

Use your calculators to calculate:

1) $\sin(45^\circ + 30^\circ)$

$= \sin(75^\circ)$

$= 0\cdot97$

2) $\sin45^\circ + \sin30^\circ$

$= 0\cdot71 + 0\cdot5$

$= 1\cdot21$

$$\sin(\alpha + \beta) \neq \sin\alpha + \sin\beta$$

Addition Formulae

Today's Learning:
By the end of this lesson, you should be able to expand functions such as $\sin(\alpha + \beta)$ and use to help solve problems.

These are given in the exam in a condensed format.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

NOTE

1) Show that $\sin(a+b) = \sin a\cos b + \cos a\sin b$

for $a = \frac{\pi}{3}$ and $b = \frac{\pi}{6}$

LHS $= \sin(a+b)$

$= \sin(\frac{\pi}{3} + \frac{\pi}{6})$

$= \sin(\frac{\pi}{2})$

$= 1$

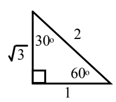
RHS $= \sin a\cos b + \cos a\sin b$

$= \sin\frac{\pi}{3}\cos\frac{\pi}{6} + \cos\frac{\pi}{3}\sin\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2}$

$= \frac{3}{4} + \frac{1}{4}$

$= 1$

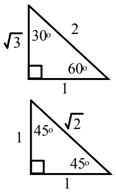


LHS = RHS as required

2) Find the exact value of $\sin 75°$.

$\sin 75° = \sin(45° + 30°)$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$



$\sin(45+30)$

$= \sin 45 \times \cos 30 + \cos 45 \times \sin 30$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

We need to rationalise the denominator

$= \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2}(\sqrt{3}+1)}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$

Starter

A function, f, is defined on the set of real numbers by $f(x) = x^3 - 7x - 6$.

Determine whether f is increasing or decreasing when $x = 2$. 3

$f'(x) = 3x^2 - 7$

$f'(2) = 3 \times 4 - 7$

$= 12 - 7$

$= 5$

increasing

3) This is a common exam question.

Acute angles p and q are such that

$\sin p = \frac{4}{5}$ and $\sin q = \frac{5}{13}$

Show that $\sin(p+q) = \frac{63}{65}$

LHS $= \sin(p+q)$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\sin(p+q) = \sin p\cos q + \cos p\sin q$

$\sqrt{13^2 - 5^2}$

$= \sqrt{169 - 25}$

$= \sqrt{144}$

$= \frac{4}{5} \times \frac{12}{13} + \frac{3}{5} \times \frac{5}{13}$

$= \frac{48}{65} + \frac{15}{65}$

$= \frac{63}{65}$

= RHS as required

now try ex 11D Q7/8

Addition Formulae (contd)

Today's Learning:
By the end of this lesson, you should be able to expand functions such as $\cos(\alpha + \beta)$ and use to help solve problems.

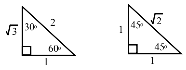
Examples

4) Find the exact value for $\cos 15°$.

$\cos 15° = \cos(45 - 30)°$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\cos(45 - 30) = \cos 45\cos 30 + \sin 45\sin 30$



$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

We need to rationalise the denominator

$= \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2}(\sqrt{3}+1)}{4}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

5)

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\cos(x° + 60°) = \cos x\cos 60° - \sin x\sin 60°$

$= \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$

Trig Identities

Learning Intention
By the end of this lesson, you should be able to use the addition formulae to help prove more complex trigonometric identites.

These are the trig identities we know from N5 Maths.

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$\cos^2 x = 1 - \sin^2 x$

$\tan x = \frac{\sin x}{\cos x}$

$\sin x = \tan x\cos x$

$\cos x = \frac{\sin x}{\tan x}$

Examples

1) Prove that $\cos(x + 90) = -\sin x$

$\text{LHS} = \cos(x + 90)$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$= \cos x \cos 90 - \sin x \sin 90$

$= 0 \times \cos x - \sin x \times 1$

$= -\sin x$

$= \text{RHS}$ as required

2) Prove that $\dfrac{\sin(x+y)}{\cos x \cos y} = \tan x + \tan y$

$\text{LHS} = \dfrac{\sin(x+y)}{\cos x \cos y}$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$= \dfrac{\sin x \cos y + \cos x \sin y}{\cos x \cos y}$

$= \dfrac{\sin x \cancel{\cos y}}{\cos x \cancel{\cos y}} + \dfrac{\cancel{\cos x} \sin y}{\cancel{\cos x} \cos y}$

$= \dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}$

You should know from N5 that $\tan x = \dfrac{\sin x}{\cos x}$

$= \tan x + \tan y$

$= \text{RHS}$ as required

$\boxed{\text{Ex 11E}}$

3) Show that $\cos(x - \frac{\pi}{6}) - \cos(x + \frac{\pi}{6}) = \sin x$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\cos(x - \frac{\pi}{6}) = \cos x \cos\frac{\pi}{6} + \sin x \sin\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x$

$\cos(x + \frac{\pi}{6}) = \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$

$\text{LHS} = \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x - (\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x)$

$= \frac{1}{2}\sin x + \frac{1}{2}\sin x$

$= \sin x$

$= \text{RHS}$ as required

Starter

When $\cos 7x° \cos 5x° + \sin 7x° \sin 5x°$ is simplified, the answer is

A. $\cos 2x°$ $\cos(7x-5x)$
B. $\cos 12x°$ $\cos(7x+5x)$
C. $\sin 2x°$ $\sin(7x-5x)$
D. $\sin 12x°$ $\sin(7x+5x)$

When $\sin(x+2y)\cos(y-x) + \cos(x+2y)\sin(y-x)$ is simplified it equals

A. $\cos(2x+y)$
B. $\sin(2x+y)$
C. $\cos 3y$
D. $\sin 3y$

$\sin\left((x+2y)+(y-x)\right)$

$\sin(3y)$

Double Angle Formulae

Today's Learning:
By the end of this lesson, you should be able to use your previous knowledge of addition formulae to determine the double angle formulae.

Expand $\sin(x + x)$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\sin(x + x) = \sin x \cos x + \cos x \sin x$

$\sin 2x = 2\sin x \cos x$

Expand $\cos(x + x)$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\cos(x + x) = \cos x \cos x - \sin x \sin x$

$\cos 2x = \cos^2 x - \sin^2 x$

You should know $\cos^2 x + \sin^2 x = 1$ from N5 and
$\sin^2 x = 1 - \cos^2 x$
$\cos^2 x = 1 - \sin^2 x$

$\cos 2x = \cos^2 x - (1 - \cos^2 x)$

$= \cos^2 x - 1 + \cos^2 x$

$\cos 2x = 2\cos^2 x - 1$

or

$\cos 2x = \cos^2 x - \sin^2 x$

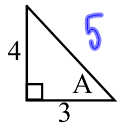
$= (1 - \sin^2 x) - \sin^2 x$

$\cos 2x = 1 - 2\sin^2 x$

You should know
$\cos^2 x + \sin^2 x = 1$
from N5
and
$\sin^2 x = 1 - \cos^2 x$
$\cos^2 x = 1 - \sin^2 x$

Examples

1) A is acute with $\tan A = \frac{4}{3}$.
Calculate $\sin 2A$ and $\cos 2A$.



Any expansion of $\cos 2A$ may be used

$\sin 2A = 2\sin A \cos A$

$= 2 \times \frac{4}{5} \times \frac{3}{5}$

$= \frac{24}{25}$

$\cos 2A = 2\cos^2 A - 1$

$= 2 \times \left(\frac{3}{5}\right)^2 - 1$

$= 2 \times \frac{9}{25} - 1$

$= \frac{18}{25} - \frac{25}{25}$

$= -\frac{7}{25}$

2) Prove $(\sin x + \cos x)^2 = 1 + \sin 2x$

LHS $= (\sin x + \cos x)^2$

$= (\sin x + \cos x)(\sin x + \cos x)$

$= \sin^2 x + \sin x\cos x + \cos x\sin x + \cos^2 x$

$= \sin^2 x + 2\sin x\cos x + \cos^2 x$

$= 1 + 2\sin x\cos x$

$= 1 + \sin 2x$

= RHS as required

3) Prove $\frac{(1 + \cos 2x)}{\sin 2x} = \frac{1}{\tan x}$

$\tan x = \frac{\sin x}{\cos x}$

$\frac{1}{\tan x} = \frac{\cos x}{\sin x}$

LHS $= \frac{(1 + \cos 2x)}{\sin 2x}$

$= \frac{(1 + \cos 2x)}{2 \times \sin x \times \cos x}$

$\cos 2x = \cos^2 x - \sin^2 x$
$= 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$

$= \frac{1 + 2\cos^2 x - 1}{2 \times \sin x \times \cos x}$

$= \frac{\cancel{2}\cos^{\cancel{2}} x}{\cancel{2} \times \sin x \times \cancel{\cos x}}$

$= \frac{\cancel{2} \times \cos x \times \cancel{\cos x}}{\cancel{2} \times \sin x \times \cancel{\cos x}}$

$= \frac{\cos x}{\sin x}$

$= \frac{\cos x}{\cos x \times \tan x}$

$= \frac{1}{\tan x}$

= RHS as required

You should know
$\tan x = \frac{\sin x}{\cos x}$
from N5
and
$\sin x = \cos x \times \tan x$
$\cos x = \frac{\sin x}{\tan x}$

Trig Equations Revision

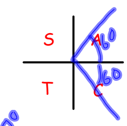
Learning Intention
By the end of this lesson, you should feel comfortable at solving trigonometric equations.

1) Solve $2\cos 3x - 1 = 0$ for $0 \leq x \leq 360^\circ$

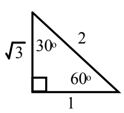
$\cos 3x = \frac{1}{2}$

$0 \leq 3x \leq 1080^\circ$

CAREFUL!!! $0 \leq x \leq 360$ has two solutions
$0 \leq 3x \leq 1080$ has six solutions



$\cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$



$3x = 60, 300, 420, 660, 780, 1020$

$x = 20, 100, 140, 220, 260, 340$

2) Solve $1 + \sqrt{2}\sin 6t = 0$ for $0 \leq t \leq 180^\circ$

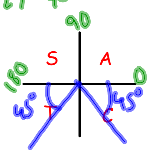
$\sqrt{2}\sin 6t = -1$

$\sin(6t) = \frac{-1}{\sqrt{2}}$

$\sin 6t = -\frac{1}{\sqrt{2}}$

$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$

CAREFUL!!! $0 \leq t \leq 180$ has one solution
$0 \leq 6t \leq 1080$ has six solutions



$6t = 225, 315, 585, 675, 945, 1035,$

$t = 37.5, 52.5, 97.5, 112.5, 157.5, 172.5$

3) Solve $2\sin(2x - \frac{\pi}{3}) = 1$ for $0 \leq x \leq 2\pi$

$0 \leq 2x \leq 4\pi$

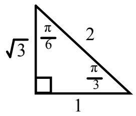
$-\frac{\pi}{3} \leq 2x - \frac{\pi}{3} \leq \frac{4\pi}{3}$

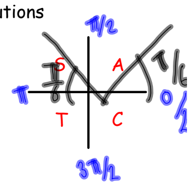
$\sin(2x - \frac{\pi}{3}) = \frac{1}{2}$

$\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$

CAREFUL!!! $0 \leq x \leq 2\pi$ has two solutions

$0 \leq 2x \leq 4\pi$ has four solutions





$2x - \frac{\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

$2x = \frac{3\pi}{6}, \frac{7\pi}{6}, \frac{15\pi}{6}, \frac{19\pi}{6}$

$x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}$

If you worked through in degrees:

$x = 45^\circ, 105^\circ, 225^\circ, 285^\circ$

We must convert our answers to radians.

Factors of 180: 1 x 180, 2 x 90, 3 x 60, 4 x 45, 5 x 36, 6 x 30, 9 x 20, 10 x 18, 12 x 15

$45^\circ = \frac{\pi}{4}$

$105^\circ = 7 \times 15 = 7 \times \frac{\pi}{12} = \frac{7\pi}{12}$

$225^\circ = 5 \times 45 = 5 \times \frac{\pi}{4} = \frac{5\pi}{4}$

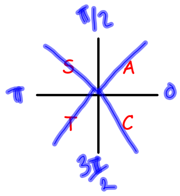
$285^\circ = 19 \times 15 = 19 \times \frac{\pi}{12} = \frac{19\pi}{12}$

4) Solve $\tan^2 x = 3$ for $0 \leq x \leq 2\pi$

$\tan x = \pm\sqrt{3}$

$\tan x = \sqrt{3}$ $\tan x = -\sqrt{3}$

$\tan^{-1}(\sqrt{3}) = \pi/3$



$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

If you worked through in degrees:

$x = 60^\circ, 120^\circ, 240^\circ, 300^\circ$

We must convert our answers to radians.

Factors of 180: 1 x 180, 2 x 90, 3 x 60, 4 x 45, 5 x 36, 6 x 30, 9 x 20, 10 x 18, 12 x 15

$60^\circ = \frac{\pi}{3}$

$120^\circ = 2 \times 60 = 2 \times \frac{\pi}{3} = \frac{2\pi}{3}$

$240^\circ = 2 \times 120 = 2 \times \frac{2\pi}{3} = \frac{4\pi}{3}$

$300^\circ = 5 \times 60 = 5 \times \frac{\pi}{3} = \frac{5\pi}{3}$

3) Solve $3\sin^2 x - 4\sin x + 4 = 3$ for $0 \leq x \leq 2\pi$

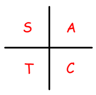
$3\sin^2 x - 4\sin x + 1 = 0$

$(3\sin x - 1)(\sin x - 1) = 0$

Either $3\sin x - 1 = 0$ or $\sin x - 1 = 0$

$\sin x = \frac{1}{3}$ $\sin x = 1$

This is not an exact value so we must use our calculator.



$x = 19{\cdot}5^\circ, 160{\cdot}5^\circ$ $x = \frac{\pi}{2}$

$x = 19{\cdot}5^\circ, \frac{\pi}{2}, 160{\cdot}5^\circ$

We must convert degrees to radians. To do this we x π then ÷ 180

$x = 0{\cdot}34, \frac{\pi}{2}, 2{\cdot}80$

4) Solve $5\sin^2 x - 2 = 2\cos x$ for $0 \leq x \leq 2\pi$

$5(1 - \cos^2 x) - 2\cos x - 2 = 0$

$5 - 5\cos^2 x - 2\cos x - 2 = 0$

$-5\cos^2 x - 2\cos x + 3 = 0$

$5\cos^2 x + 2\cos x - 3 = 0$

$(5\cos x - 3)(\cos x + 1) = 0$

Either $5\cos x - 3 = 0$ or $\cos x + 1 = 0$

$\cos x = \frac{3}{5}$ $\cos x = -1$

$x = \pi$

This is not an exact value so we must use our calculator.

$x = 53{\cdot}13^\circ, 306{\cdot}87^\circ$

$x = 53{\cdot}13^\circ, \pi, 306{\cdot}87^\circ$

We must convert degrees to radians. To do this we x π then ÷ 180

$x = 0{\cdot}93, \pi, 5{\cdot}36$

Further Trigonometric Equations

Today's Learning:
By the end of this lesson, you should be able to use the Double Angle Formulae to solve Trig Equations.

We can use the Double Angle Formulae to help us solve Trig Equations.
You have seen basic Trig Equations before in N5 and Unit 1.

Examples

1) Solve $\sin 2x = -\sin x$ for $0 \leq x \leq 360^\circ$

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$

$2\sin x\cos x = -\sin x$

$2\sin x\cos x + \sin x = 0$

$\sin x(2\cos x + 1) = 0$

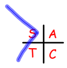
Either $\sin x = 0$

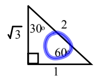
$x = 0, 180, 360$

or $2\cos x + 1 = 0$

$\cos x = -\frac{1}{2}$

$x = 120, 240$





$$\underline{\underline{x = 0^\circ, 120^\circ, 180^\circ, 240^\circ, 360^\circ}}$$

2) Solve $\cos 2x - 4\sin x + 5 = 0$ for $0 \leq x \leq 2\pi$

Which expansion for $\cos 2x$ will we use?

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$

$1 - 2\sin^2 x - 4\sin x + 5 = 0$

$-2\sin^2 x - 4\sin x + 6 = 0$

$2\sin^2 x + 4\sin x - 6 = 0$

$2(\sin^2 x + 2\sin x - 3) = 0$

$2(\sin x - 1)(\sin x + 3) = 0$

Either $\sin x - 1 = 0$ or $\sin x + 3 = 0$

$\sin x = 1$ $\quad$ $\sin x = -3$

not poss

$x = \frac{\pi}{2}$

$$\underline{\underline{x = \frac{\pi}{2}}}$$

3) Solve $\cos 2x = \cos x$ for $0 \leq x \leq 2\pi$

Which expansion for $\cos 2x$ will we use?

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$

$2\cos^2 x - 1 = \cos x$

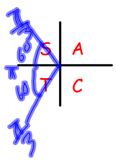
$2\cos^2 x - \cos x - 1 = 0$

$(2\cos x + 1)(\cos x - 1) = 0$

Either $2\cos x + 1 = 0$ or $\cos x - 1 = 0$

$\cos x = -\frac{1}{2}$ $\quad$ $\cos x = 1$

$x = 0, 2\pi$



$x = \frac{2\pi}{3}, \frac{4\pi}{3}$

$$\underline{\underline{x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi}}$$

Starter

Given that a is an acute angle and that $\tan a = \frac{1}{5}$, the exact value of $\sin 2a$ is

A. $\frac{1}{13}$

B. $\frac{5}{12}$

C. $\frac{5}{13}$

D. $\frac{1}{12}$

$\sin(2a) = 2\sin a\cos a$

$= \frac{2}{1} \times \frac{1}{\sqrt{26}} \times \frac{5}{\sqrt{26}}$

$= \frac{10}{26}$

$= \frac{5}{13}$

Starter

1) Change the subject of the formula to $\cos^2 x$.

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x + 1 = 2\cos^2 x$

$\cos^2 x = \frac{\cos 2x + 1}{2}$

2) Change the subject of the formula to $\sin^2 x$.

$\cos 2x = 1 - 2\sin^2 x$

$\frac{\cos 2x - 1}{-2} = \sin^2 x$

$\frac{1 - \cos 2x}{2} = \sin^2 x$

Formulae for $\cos^2 x$ and $\sin^2 x$

Today's Learning:
By the end of this lesson you should be able to use the formulae for $\cos^2 x$ and $\sin^2 x$ to help prove trig identities.

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

Example

1) Express $2\cos^2 x - 3\sin^2 x$ in terms of $\cos 2x$.

$$2\cos^2 x - 3\sin^2 x$$
$$= 2 \times \frac{1}{2}(1 + \cos 2x) - 3 \times \frac{1}{2}(1 - \cos 2x)$$
$$= (1 + \cos 2x) - \frac{3}{2}(1 - \cos 2x)$$
$$= 1 + \cos 2x - \frac{3}{2} + \frac{3}{2}\cos 2x$$
$$= \frac{5}{2}\cos 2x - \frac{1}{2}$$

Ex 11I Q1 -4