2009 Applied Mathematics

Advanced Higher – Statistics

Finalised Marking Instructions

Advanced Higher Applied Mathematics 2009
Statistics Solutions

A1. (a) The upper and lower chart limits are given by:

$$\bar{x} \pm 3\frac{\sigma}{\sqrt{n}} = 50 \pm 3 \times \frac{0{\cdot}4}{\sqrt{4}} = 49{\cdot}4,\ 50{\cdot}6 \qquad \mathbf{1}$$

The probability that a point breaches the limits is $\mathrm{P}(Z > 3 \text{ or } Z < -3)$ **1**

$= 2(0{\cdot}0013) = 0{\cdot}0026$ **1**

(b) The probability that a point now plots outwith the limits is

$$\mathrm{P}\left(Z > \frac{50{\cdot}6 - 50{\cdot}5}{0{\cdot}2}\right) + \mathrm{P}\left(Z < \frac{49{\cdot}4 - 50{\cdot}5}{0{\cdot}2}\right)$$

$$= \mathrm{P}(Z > 0{\cdot}5) + \mathrm{P}(Z < -5{\cdot}5) \qquad \mathbf{1}$$

$$= 0{\cdot}3085 + 0 = 0{\cdot}3085 \qquad \mathbf{1}$$

We would thus expect 3 'out of controls' in 10 samples **1**

or 3 times in 150 minutes i.e. every 50 minutes. **1**

A2. (a) The sample mean is $413{\cdot}4/9 = 45{\cdot}93$. **1**

A 95% confidence interval is given by

$$\bar{x} \pm 1{\cdot}96\frac{\sigma}{\sqrt{n}} \qquad \mathbf{1}$$

$$= 45{\cdot}93 \pm 1{\cdot}96 \times \frac{6}{\sqrt{9}}$$

$$= (42{\cdot}0, 49{\cdot}8) \qquad \mathbf{1}$$

(b) 24/25 = 96% of intervals capture $\mu = 45$. **1**

This is close to the expected capture rate of 95%. **1**

A3. (a) The product moment correlation coefficient is

$$r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$$

$$= \frac{55{\cdot}01}{\sqrt{111{\cdot}00 \times 41{\cdot}27}} = 0{\cdot}816 \qquad \mathbf{1}$$

$H_0: \rho = 0 \quad H_1: \rho \neq 0$ **1**

$$t = \frac{r}{\sqrt{\frac{1-r^2}{n-2}}} = \frac{0{\cdot}816}{\sqrt{\frac{1-0{\cdot}816^2}{11-2}}}$$

$$= 4{\cdot}23 \qquad \mathbf{1}$$

The critical region for 9df at 1% level is $|t| > 3{\cdot}25$. **1**

Since 4·23 lies in the critical region, the null hypothesis that $\rho = 0$ would be rejected at the 1% level. **1**

(b) The correlation coefficient is an appropriate statistic for the first data set but not for the others. **1**

A4. (a) H_0: There is no association between serum cholesterol level and the presence or absence of heart disease.
H_1: There is an association. **1**

Since the p-value is less than 0·01 the null hyothesis would be rejected at the 1% level so the data provide **1**
strong evidence of an association between serum cholesterol level and the presence or absence of heart disease. **1**

(b) The expected frequencies are bracketed in the table.

	Present	Absent
< 7.00	28 (36.99)	425 (416.01)
≥ 7.00	21 (12.01)	126 (135.00)

1

$$x^2 = \sum \frac{(O - E)^2}{E}$$

$$= \frac{(28 - 36{\cdot}99)^2}{36{\cdot}99} + \frac{(425 - 416{\cdot}01)^2}{416{\cdot}01} + \frac{(21 - 12{\cdot}01)^2}{12{\cdot}01} + \frac{(126 - 135{\cdot}00)^2}{135{\cdot}00}$$

$$= 2{\cdot}185 + 0{\cdot}194 + 6{\cdot}729 + 0{\cdot}600 = 9{\cdot}708$$ **1**

with 1 d.f.
Since 9·708 lies between 7·879 and 10·827 **1**
The p-value lies in the interval (0·001, 0·005). **1**

A5. (a) The number of dogs that benefit is given by $D \sim \mathrm{B}(100,\ 0{\cdot}8)$ **1**
which can be approximated by N(80, 16). **1**

$$P(\text{Claim rejected}) = P(D \leqslant 74)$$

$$\approx P\left(Z \leqslant \frac{74{\cdot}5 - 80}{4}\right)$$ **1**

$$\approx 0{\cdot}0838$$ **1**

(b) B(100, 0·7) approximated by N(70, 21)

$$P(\text{Claim rejected}) = P(D \geqslant 75)$$

$$\approx P\left(Z \geqslant \frac{74{\cdot}5 - 70}{4{\cdot}58}\right)$$ **1**

$$\approx 0{\cdot}1635$$ **1**

A6. (a) Proportion of deficient documents is 3/10= 0·3. **1**

$$\bar{x} = \frac{0+1+0+0+1+0+0+0+0+1}{10} = \frac{3}{10} = 0·3$$ **1**

(b) Sample proportion = $\frac{1}{n}X_1 + \frac{1}{n}X_2 + \frac{1}{n}X_3 + \dots + \frac{1}{n}X_n$.

E(Sample proportion)

$$= \frac{1}{n}\,\mathrm{E}(X_1) + \frac{1}{n}\,\mathrm{E}(X_2) + \frac{1}{n}\,\mathrm{E}(X_3) + \dots + \frac{1}{n}\,\mathrm{E}(X_n)$$ **1**

$$= \frac{1}{n}p + \frac{1}{n}p + \frac{1}{n}p + \dots + \frac{1}{n}p = n \times \frac{1}{n}p = p$$ **1**

V(Sample proportion)

$$= \frac{1}{n^2}\,\mathrm{V}(X_1) + \frac{1}{n^2}\,\mathrm{V}(X_2) + \frac{1}{n^2}\,\mathrm{V}(X_3) + \dots + \frac{1}{n^2}\,\mathrm{V}(X_n)$$ **1**

$$= \frac{1}{n^2}pq + \frac{1}{n^2}pq + \frac{1}{n^2}pq + \dots + \frac{1}{n^2}pq$$

$$= n \times \frac{1}{n^2}pq = \frac{pq}{n}$$ **1**

(c) Since a sample proportion may be regarded as a sample mean
the central limit theorem indicates that a sample proportion
will be approximately normally distributed when the sample size is large. **1**

A7. (a)

Ht.	Co.	Rank
147	A	1
149	A	2
151	B	3
153	A	4
155	A	5
157	B	6
159	B	7
163	B	8
165	B	9
169	B	10

1

Rank sum for A is $W = 1 + 2 + 4 + 5 = 12$ **1**

(b) Number of potential subsets

$$^{10}C_4 = 210$$ **1**

(c) {1,2,3,4} {1,2,3,5} {1,2,3,6} {1,2,4,5} **1,1**

P(Rank sum for A is less than or equal to 12)

= 4/210 = 2/105 **1**

(d) Since 2/105 is less than 0.05 the null hypothesis would **1**
be rejected in favour of the alternative at the 5% level **1**
thus furnishing evidence that the B plants appear to grow
taller than the A plants. **1**

A8. (a) Let F denote battery failure during warranty period.

$P(A \mid F)$ is required.

$$P(A \mid F) = \frac{P(F \cap A)}{P(F)} = \frac{P(A \cap F)}{P(F)} \quad \mathbf{1}$$

$$= \frac{P(A)P(F \mid A)}{P(A)P(F \mid A) + P(B)P(F \mid B) + P(C)P(F \mid C)} \quad \mathbf{1}$$

$$= \frac{0{\cdot}6 \times 0{\cdot}03}{0{\cdot}6 \times 0{\cdot}03 + 0{\cdot}3 \times 0{\cdot}01 + 0{\cdot}1 \times 0{\cdot}2} \quad \mathbf{1}$$

$$= \frac{0{\cdot}018}{0{\cdot}018 + 0{\cdot}003 + 0{\cdot}002} = \frac{0{\cdot}018}{0{\cdot}023} = \frac{18}{23} \quad \mathbf{1}$$

{*Alternative methods such as Venn or tree diagrams are acceptable*}

(b) $\mathrm{B}\left(5, \frac{18}{23}\right)$ **1**

$$\mathrm{P}(B = 3) = {}^5C_3\left(\frac{18}{23}\right)^3\left(\frac{5}{23}\right)^2 \quad \mathbf{1}$$

$$= 0{\cdot}2265 \quad \mathbf{1}$$

(c)

$$P(B \mid F) = \frac{0{\cdot}003}{0{\cdot}018 + 0{\cdot}003 + 0{\cdot}002} = \frac{0{\cdot}003}{0{\cdot}023} \approx \frac{3}{23}$$

$$P(C \mid F) = \frac{0{\cdot}002}{0{\cdot}018 + 0{\cdot}003 + 0{\cdot}002} = \frac{0{\cdot}002}{0{\cdot}023} \approx \frac{2}{23} \quad \mathbf{1}$$

A should be allocated 0·783 × 200000 = 156600 (156522) **1**

B should be allocated 0·130 × 200000 = 26000 (26087)

C should be allocated 0·087 × 200000 = 17400 (17391) **1**

A9. (a) An apparent superior performance by one type of tyre might be due to differences between drivers and not the tyres. **1**

(b) The essential assumption is that the differences are normally distributed. **1**

The mean and standard deviation of the differences are 0·032 and 0.026.

$H_0: \mu_d = 0 \quad H_1: \mu_d \neq 0$ **1**

$$t = \frac{\bar{d} - \mu_d}{\frac{s_d}{\sqrt{n}}} = \frac{0{\cdot}032 - 0}{\frac{0{\cdot}026}{\sqrt{10}}} \quad \mathbf{1}$$

$$= 3{\cdot}89 \quad \mathbf{1}$$

The critical region at the 1% level of significance with 9 degrees of freedom is $|t| > 3{\cdot}25$ **1**

Since 3·89 exceeds 3·25 the null hypothesis is rejected at the 1% level so the data provide strong evidence of different rates of wear for the two types of tyre. **1**

(c) Of the 9 non-zero differences only one is positive. **1**

$\mathrm{P}(X \leqslant 1 \mid X \sim \mathrm{B}(9,\ 0{\cdot}5)) = (1 + 9)0{\cdot}5^9 = 0{\cdot}0195$ **1**

The p-value is therefore $2 \times 0{\cdot}0195 = 0{\cdot}0390$ **1**

Since the 0·01 < 0·0390 < 0·05, the sign test provides evidence, at only the 5% level, of different rates of wear for the two types of tyre, unlike the *t*-test which provides evidence at the 1% level. **1**

[END OF STATISTICS SOLUTIONS]

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Section B Solutions

B1.

$$\left(b - \frac{2}{b}\right)^5 = b^5 + 5b^4\left(-\frac{2}{b}\right) + 10b^3\frac{4}{b^2} + 10b^2\left(-\frac{8}{b^3}\right) + 5b\frac{16}{b^4} - \frac{32}{b^5}$$ **powers 1** **coeffs 1** **signs 1**

$$= b^5 - 10b^3 + 40b - \frac{80}{b} + \frac{80}{b^3} - \frac{32}{b^5}$$ **1**

B2.

$$u = \cos x \Rightarrow du = -\sin x\,dx,$$ **1**

$$x = 0 \Rightarrow u = 1; \qquad x = \tfrac{\pi}{3} \Rightarrow u = \tfrac{1}{2}$$ **1**

Hence

$$\int_0^{\pi/3} \cos^5 x \sin x\,dx = -\int_1^{\frac{1}{2}} u^5 du = \left[-\frac{1}{6}u^6\right]_1^{\frac{1}{2}}$$ **1**

$$= -\frac{1}{6}\frac{1}{64} + \frac{1}{6} = \frac{21}{128} (\approx 0.164)$$ **1**

OR

$$\int_0^{\pi/3} \cos^5 x \sin x\,dx = \left[-\frac{1}{6}\cos^6 x\right]_0^{\pi/3}$$ **3E1**

$$= -\frac{1}{6}\frac{1}{64} + \frac{1}{6} = \frac{21}{128} (\approx 0.164)$$ **1**

B3.

$$x = t^2 + 1 \Rightarrow \frac{dx}{dt} = 2t$$

$$y = 1 - 3t^3 \Rightarrow \frac{dy}{dt} = -9t^2$$ **1**

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$ **M1**

$$= \frac{-9t^2}{2t} = \frac{-9t}{2}$$

$$= -9 \text{ when } t = 2.$$ **1**

Point of contact is $x = 5, y = -23$. **1**

Equation of tangent is

$$(y + 23) = -9(x - 5)$$ **1**

$$y + 23 = -9x + 45$$

$$y + 9x = 22$$

B4.

$$\det\begin{pmatrix} 1 & 1 & 0 \\ 0 & k-2 & -1 \\ 1 & 2 & k \end{pmatrix} = 1\det\begin{pmatrix} k-2 & -1 \\ 2 & k \end{pmatrix} - 1\det\begin{pmatrix} 0 & -1 \\ 1 & k \end{pmatrix} + 0 \qquad \textbf{M1,1}$$

$$= (k-2)k + 2 - (0+1) \qquad \textbf{1}$$

$$= k^2 - 2k + 1 = (k-1)^2 = 0.$$

Hence the matrix does not have an inverse when $k = 1$. **1**

B5.

$$t\frac{dx}{dt} - 2x = 3t^2$$

$$\frac{dx}{dt} - \frac{2}{t}x = 3t \qquad \textbf{1}$$

Integrating factor: $\int -\frac{2}{t}dt = -2\ln t = \ln t^{-2}$ so IF = t^{-2}. **M1,1**

$$\frac{1}{t^2}\frac{dx}{dt} - \frac{2}{t^3}x = \frac{3}{t}$$

$$\frac{x}{t^2} = \int \frac{3}{t}dt \qquad \textbf{1}$$

$$= 3\ln t + c$$

$$x = t^2(3\ln t + c) \qquad \textbf{1}$$

$$(1,1) \Rightarrow c = 1 + 0$$

$$x = t^2(1 + 3\ln t) \qquad \textbf{1}$$

B6.

$$f(x) = x\tan 2x$$

$$f'(x) = \tan 2x + 2x\sec^2 2x \qquad \textbf{M1,1}$$

$$f''(x) = 2\sec^2 2x + 2\sec^2 2x + 2x(4\sec 2x(\sec 2x\tan 2x)) \qquad \textbf{2E1}$$

$$= 4\sec^2 2x + 8x\sec^2 2x\tan 2x \qquad \textbf{1}$$

$$= 4\sec^2 2x(1 + 2x\tan 2x).$$

$$\int_0^{\pi/6} \frac{1 + 2x\tan 2x}{\cos^2 2x}\,dx = \frac{1}{4}\int_0^{\pi/6} 4\sec^2 2x(1 + 2x\tan 2x)\,dx \qquad \textbf{1,1}$$

$$= \frac{1}{4}\left[\tan 2x + 2x\sec^2 2x\right]_0^{\pi/6} \qquad \textbf{1}$$

$$= \frac{1}{4}\left[\sqrt{3} + \frac{\pi}{3}2^2\right] \qquad \textbf{1}$$

$$= \frac{\sqrt{3}}{4} + \frac{\pi}{3}.$$

[END OF SECTION B SOLUTIONS]