Solutions

Ex 3.4

1) $f(x) = \begin{cases} \frac{1}{5} & 0 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$

a)

b) *Uniform Distribution*

$E(X) = \frac{1}{2}(0+5) = 2.5$

$Var(X) = \frac{(5-0)^2}{12} = \frac{25}{12} = 2.08\dot{3}$

c) $n = 25$ As $n > 20$ by CLT

$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$\mu = E(X) = 2.5$

$\sigma^2 = Var(X) = 2.08\dot{3}$ $\frac{\sigma^2}{n} = 0.08\dot{3}$

so $\bar{X} \sim N(2.5, 0.08\dot{3})$

d) i) $P(X < 2) = P(X = 0) + P(X = 1)$

$$= \frac{1}{5} + \frac{1}{5}$$

$$= \frac{2}{5} \text{ or } 0.4.$$

ii) $P(\bar{X} < 2)$

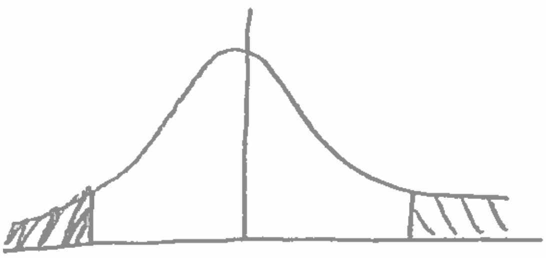
$\bar{X} \sim N(2.5, 0.08\dot{3})$

$$P\left(\bar{X} < \frac{2 - 2.5}{\sqrt{0.08\dot{3}}}\right) = P(Z < -1.732)$$

$$= 1 - P(Z < 1.732)$$

$$= 1 - 0.9582$$

$$= \underline{\underline{0.0418}}$$



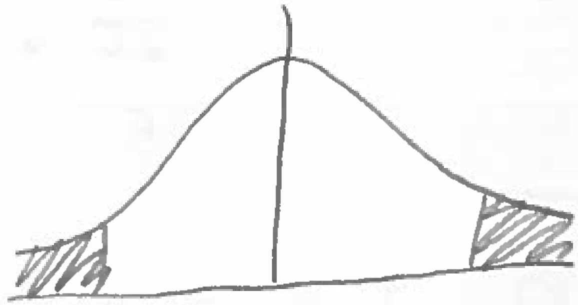
2) As $n = 50$, greater than 20, by CLT

$$\bar{X} \sim N\left(10, \frac{100}{50}\right)$$

$$\bar{X} \sim N(10, 2)$$

b) i) $P(\bar{X} < 8)$

$$= P\left(\bar{X} < \frac{8-10}{\sqrt{2}}\right)$$



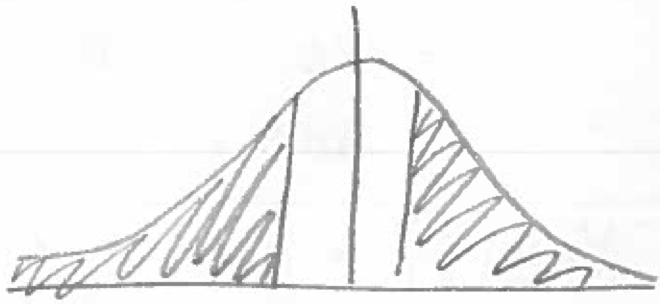
$$= P(Z < -1.414)$$

$$= 1 - P(Z < 1.414)$$

$$= 1 - 0.9207$$

$$= 0.0793$$

ii) $P(\bar{X} > 11)$



$$= P\left(\bar{X} > \frac{11-10}{\sqrt{2}}\right)$$

$$= P(\bar{X} > 0.707)$$

$$= 1 - P(\bar{X} < 0.707)$$

$$= 1 - 0.7611$$

$$= 0.2389$$

3) $X_i \sim B(20, 0.5)$

$E(X_i) = np = 20 \times 0.5$

$E(X_i) = 10$

$Var(X_i) = npq$

$= 20 \times 0.5 \times 0.5$

$= 5.$

b) $T = \sum_{i=1}^{30} X_i$

$E(T) = E(X_1 + X_2 + \dots + X_{30})$

$= E(X_1) + E(X_2) + \dots + E(X_{30})$

$= 10 + 10 + \dots + 10$

$= 300$

$Var(T) = Var(X_1 + X_2 + \dots + X_{30})$

$= Var(X_1) + Var(X_2) + \dots + Var(X_{30})$

$= 5 + 5 + \dots + 5$

$= 150$

By CLT $(n = 30)$

$T \sim N(300, 150)$

3c) $P(T \leq 300)$ by CC $= P(T \leq 300.5)$

$$= P\left(Z \leq \frac{300.5 - 300}{\sqrt{150}}\right) = P(Z \leq 0.04)$$

$$= 0.5160$$

d) $P(\bar{X} \leq 10)$ $\quad \bar{X} = \frac{1}{30} T$

$$E(\bar{X}) = E\left(\frac{T}{30}\right) = \frac{1}{30} E(T)$$

$$= \frac{1}{30} \times 300 = 10$$

$$Var(\bar{X}) = Var\left(\frac{T}{30}\right) = \frac{1}{30^2} Var(T)$$

$$= 0.1\dot{6}$$

$\bar{X} \sim N(10, 0.1\dot{6})$

$P(\bar{X} \leq 10)$ by C.C $= P(\bar{X} \leq 10 + \frac{1}{60})$

$$= P\left(Z \leq \frac{10.01\dot{6} - 10}{\sqrt{0.1\dot{6}}}\right) = P(Z \leq 0.04)$$

$$= 0.5160$$

as in part c.

4) $f(x) = \begin{cases} \frac{1}{450}(30-x) & 0 < x < 30 \\ 0 & \text{otherwise.} \end{cases}$

$$E(X) = \int_a^b x f(x)\,dx$$

$$= \int_0^{30} \frac{x}{450}(30-x)\,dx$$

$$= \frac{1}{450}\int_0^{30} 30x - x^2\,dx$$

$$= \frac{1}{450}\left[15x^2 - \frac{x^3}{3}\right]_0^{30}$$

$$= \frac{1}{450}\left[13500 - 9000\right]$$

$$= 10.$$

$$E(X^2) = \int_a^b x^2 f(x)\,dx$$

$$= \frac{1}{450}\int_0^{30} 30x^2 - x^3\,dx$$

$$= \frac{1}{450}\left[10x^3 - \frac{x^4}{4}\right]_0^{30}$$

$$= \frac{1}{450}\left[270000 - 202500\right]$$

$$= 150$$

$$Var(X) = E(X^2) - E(X)^2$$

$$= 150 - 10^2$$

$$= \underline{\underline{50}}$$

b) $n > 20$ so CLT applies.

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\bar{X} \sim N\left(10, \frac{50}{25}\right)$$

$$\bar{X} \sim N(10, 2)$$

c) To Assume Independence.

d) ii) $P(\bar{X} < 7)$

$$= P\left(Z < \frac{7-10}{\sqrt{2}}\right)$$

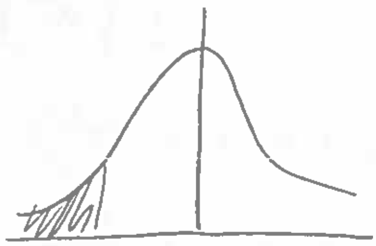
$$= P(Z < -2.12)$$

$$= P(Z > 2.12)$$

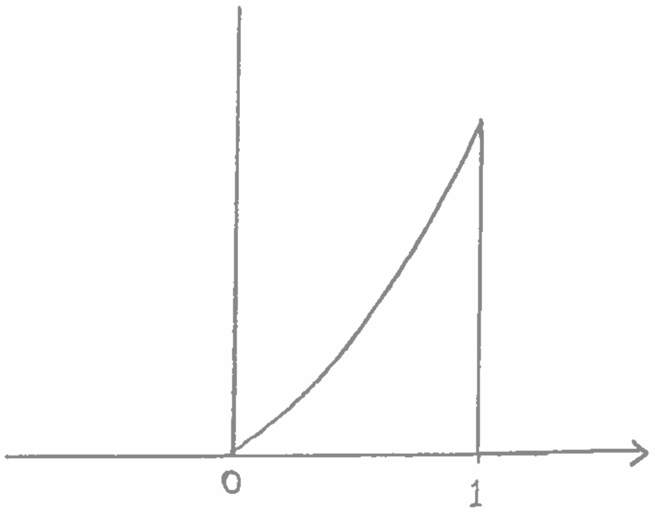
$$= 1 - P(Z < 2.12)$$

$$= 1 - 0.9830$$

$$= 0.0170.$$



5) $f(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$



b) $E(X) = \int_0^1 3x^3 \, dx$

$= \left[\frac{3}{4}x^4\right]_0^1 = \frac{3}{4}$

$Var(X) = E(X^2) - E(X)^2$

$E(X^2) = \int_0^1 3x^4 \, dx$

$= \left[\frac{3}{5}x^5\right]_0^1 = \frac{3}{5}$

$Var(X) = \frac{3}{5} - \left(\frac{3}{4}\right)^2$

$= 0.0375$

c) As $n > 20$ by CLT

$\bar{X} \sim N(0.75, 0.001)$

d) To Assume Independence.

7) a) $E(X) = (0 \times 0.6) + (1 \times 0.4)$

$= 0.4$

$E(X^2) = (0^2 \times 0.6) + (1^2 \times 0.4)$

$= 0.4$

$Var(X) = 0.4 - (0.4)^2$

$= 0.24$

b) $T = \sum_{i=1}^{25} X_i$

$E(T) = E(X_1 + X_2 + \ldots + X_{25})$

$= E(X_1) + E(X_2) + \ldots + E(X_{25})$

$= 0.4 + 0.4 + \ldots + 0.4$

$= 10$

$Var(T) = Var(X_1 + X_2 + \ldots + X_{25})$

$= Var(X_1) + Var(X_2) + \ldots + Var(X_{25})$

$= 0.24 + 0.24 + \ldots + 0.24$

$= 6$

By CLT $T \sim N(10, 6)$

$P(T \leq 15)$ by C.C. $P(T \leq 15.5)$

$P\left(Z \leq \frac{15.5 - 10}{\sqrt{6}}\right) = P(Z \leq 2.245)$

$= 0.9878$

e) i) $P(X < 0.6) = \int_0^{0.6} 3x^2 \, dx$

$= \left[x^3 \right]_0^{0.6}$

$= 0.216.$

ii) $P(\bar{X} < 0.6) = P\left(Z < \frac{0.6 - 0.75}{\sqrt{0.001}}\right)$

$= P(Z < -4.743)$

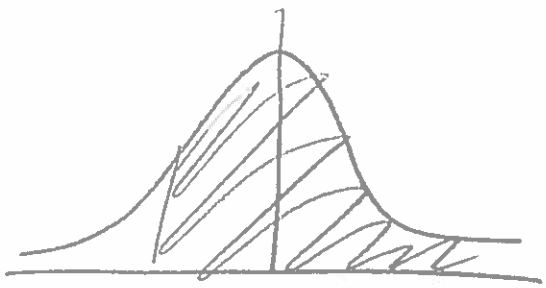
$= 1 - P(Z < 4.743)$

$= 1 - 1$

$= 0$

6) a) If $n = 30$ by CLT

$\bar{X} \sim N\left(175, \frac{400}{30}\right)$



b) $P(\bar{X} > 180)$

$= P\left(Z > \frac{180 - 175}{\sqrt{\frac{400}{30}}}\right) = P(Z > -1.37)$

$= 1 - P(Z < 1.37)$

$= 1 - 0.9147$

$= 0.0853$

7c) $W \sim B(25, 0.4)$

$P(W \leq 15)$ $\left(np = 10 \quad nq = 15 \quad \text{both} > 5\right)$

by normal approximation

$W \approx N(10, 6)$

$P(W \leq 15)$ by C.C. $= P(W \leq 15 + 0.5)$

$= P(W \leq 15.5)$

$= P\left(Z \leq \frac{15.5 - 10}{\sqrt{6}}\right) = P(Z \leq 2.245)$

$= 0.9878$

as before in part b.