Random Variables

Today we are learning...

How we represent random variables as a probability distribution.

I will know if I have been successful if...

I understand what a random variable is and give its range space.

I understand what a probability distribution is.

I can calculate the mean and variance of a discrete random variable.

Random Variables

Definition:

A random variable is a **function** which associates a unique real value with each outcome in the **sample space** of a random experiment.

(Textbook)

or...

A random variable is a **variable** whose possible values are outcomes of a random phenomenon or experiment.

(Wikipedia)

Random Variables

Consider an experiment where two dice are rolled.

The sample space is given in the textbook. (Page 33)

Suppose the random variable is the number of sixes showing on the dice.

The **range space** of X is therefore $\{x_1, x_2, x_3\} = \{0, 1, 2\}$.

In this case X is called a **discrete** random variable.

With each value of x we may associate a probability of its occurrence. $P(X = x) = p(x)$

X =

Random Variables

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Probability Distributions

With each value of x we may associate a probability of its occurrence. $P(X = x) = p(x)$

x			
p(X = x)			

Notice that if we add all of the probabilities up we get 1.

Probability Distribution

Sometimes probability distributions may look a little different.

The discrete random variable W has the distribution function given by...

$$p(w) = \begin{cases} 0.6 - 0.2w & \text{for } w = 1, 2, 3 \\ 0.4 & \text{for } w = 4 \\ 0 & \text{otherwise} \end{cases}$$

What would this look like if it were in table form?

$$p(w) = \begin{cases} 0.6 - 0.2w & \text{for } w = 1, 2, 3 \\ 0.4 & \text{for } w = 4 \\ 0 & \text{otherwise} \end{cases}$$

w				
P(W = w)				

Do the probabilities add up to 1?

Your task...

An experiment consists of rolling two dice.

Suppose the Y random variable is the **sum** of the two dice.

Using a table draw the probability distribution of Y.

Use the sample space shown on page 33 to help you.

Challenge: Can you draw this in the alternative form using a bracket?

y	2	3	4	5	6	7	8	9	10	11	12
P(Y=y)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Cumulative Distribution Function

A cumulative distribution function of a discrete random variable gives us information about $P(X \leq x)$.

Previously a probability distribution only gave us information about $P(X = x)$.

Going back to our random variable X, the number of sixes showing on the dice we came up with this probability distribution.....

Probability Distribution

x	0	1	2
$P(X = x)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

Cumulative Distribution Function

x	0	1	2
$P(X \leq x)$			

$P(X \leq 0)$ $P(X \leq 1)$ $P(X \leq 2)$

Your Task

We previously defined Y to be the sum of the two dice.

Draw a table to represent the cumulative distribution function for Y.

Use your cumulative distribution function to find....

a) The probability that the sum of two dice when rolled is less than or equal to 9.

b) The probability that the sum of two dice when rolled is strictly greater than 8.

y	2	3	4	5	6	7	8	9	10	11	12
$P(Y \leq y)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{15}{36}$	$\frac{21}{36}$	$\frac{26}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	$\frac{35}{36}$	1

Mean & Variance

The mean (denoted μ or E(X)) of the discrete random variable X is the average value that would be recorded in if an experiment was carried out.

$$\mu = E(X) = \sum_{\text{all } x} x\, p(x)$$

Calculate E(Y) for the sum of two dice.

What would we expect the sum of two dice to be?

Mean & Variance

The variance, V(X), of a discrete random variable is a measure of how spread out the recorded values of X would be if the experiment were carried out a large number of times.

$$V(X) = E(X^2) - ((E(X))^2)$$

$$\text{where } E(X^2) = \sum_{\text{all } x} x^2 p(x)$$

Calculate the V(Y) for the sum of the two dice.

Page 34 - Exercise 2.1A

Page 36 - Exercise 2.1B

The Laws of Expectation & Variance

Today we are learning...

What are the laws of expectation and variance.

I will know if I have been successful if...

I can recall the law for adding a constant.

I can recall the law for multiplying by a constant.

I can recall the law for adding and subtracting random variables.

Example 1

Let the random variable X be the score on a die when rolled. The sample space for this experiment is simply {1, 2, 3, 4, 5, 6}.

Let's calculate the E(X) and Var(X).

Use the two formula from last time.

$$\mu = E(X) = \sum_{\text{all } x} x\, p(x)$$

$$V(X) = E(X^2) - ((E(X)^2)$$

$$\text{where } E(X^2) = \sum_{\text{all } x} x^2\, p(x)$$

Let's now create a new random variable W.

Suppose W = X + 3

w						
P(W = w)						

Now let's calculate E(W) and Var(W)

$$\mu = E(X) = \sum_{\text{all } x} x\, p(x)$$

$$V(X) = E(X^2) - ((E(X)^2)$$

$$\text{where } E(X^2) = \sum_{\text{all } x} x^2 p(x)$$

What have we found?

E(X) = 3.5

$$Var(X) = \frac{35}{12}$$

Let W = X + 3

E(W) = 6.5

$$Var(W) = \frac{35}{12}$$

Law 1 - Adding a Constant

$$E(X + b) = E(X) + b$$

$$Var(X + b) = Var(X)$$

Example 2

Let the random variable X be the score on a die when rolled.

The sample space for this experiment is simply {1, 2, 3, 4, 5, 6}.

$$E(X) = 3.5$$

$$Var(X) = \frac{35}{12}$$

This time we are going to create a new random variable W, such that W = 2X.

Or in other words, double the score on the dice.

$W = 2X$

w						
P(W = w)						

Now let's calculate E(W) and Var(W)

$$\mu = E(X) = \sum_{\text{all } x} x\, p(x)$$

$$V(X) = E(X^2) - ((E(X))^2)$$

$$\text{where } E(X^2) = \sum_{\text{all } x} x^2\, p(x)$$

What have we found?

$E(X) = 3.5$

$Var(X) = \frac{35}{12}$

Let $W = 2X$

$E(W) = 7$

$Var(W) = \frac{35}{3}$

Law 2 - Multiplying by a Constant

$$E(aX) = aE(X)$$

$$Var(aX) = a^2 Var(X)$$

Example 3 - Adding a Variable.

Assume the random variable X is the score on the die.

Assume the random variable Y is score obtained on a square spinner with four sections.

The sample space is shown for X + Y is shown at the bottom of page 38.

x + y									
p(x + y)									

Now let's calculate E(X + Y) and Var(X + Y)

$$\mu = E(X) = \sum_{\text{all } x} x\, p(x)$$

$$V(X) = E(X^2) - ((E(X)^2)$$

$$\text{where } E(X^2) = \sum_{\text{all } x} x^2 p(x)$$

What have we found?

$E(X) = 3.5$

$Var(X) = \frac{35}{12}$

$E(Y) = 2.5$

$Var(Y) = \frac{15}{2}$

$E(X + Y) = 6$

$Var(X + Y) = \frac{25}{6}$

Law 3 - Adding a Variable

$E(X + Y) = E(X) + E(Y)$

$Var(X + Y) = Var(X) + Var(Y)$

Note the result for the variance is only true if X and Y are independent.

Rule 4 - Subtracting Variables

Now we are going to consider X - Y, where X is the random variable representing the score on the first die and Y the score on the spinner.

The sample space for this experiment is given on page 40.

We have already shown that...

$E(X) = 3.5$

$Var(X) = \frac{35}{12}$

$E(Y) = 2.5$

$Var(Y) = \frac{15}{2}$

x - y									
p(x - y)									

Rule 4 - Subtracting Variables

$$E(X - Y) = E(X) - E(Y)$$

$$Var(X - Y) = Var(X) + Var(Y)$$

Again this is only true for the variance if the X and Y are independent.

A summary of all four rules are given at the top of page 41.

Starter

Complete question 5 & 6 from Page 41

Binomial Distribution

Today we are learning...

What the binomial distribution is and how we use it.

I will know if I have been successful if...

I know what the Binomial Distribution is.

I understand the conditions that must be met to use it.

I can calculate probabilities using it.

The Binomial Distribution

The binomial distribution is a special discrete probability distribution.

If X~Bin(n, p) then:

$$P(X = x) = \begin{cases} \binom{n}{x} p^x q^{n-x} & \text{for } x = 0, 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

where $0 < p < 1$ and $q = 1 - p$

Binomial Distribution

The Binomial Distribution can be used to calculate probabilities of certain experiments. The following conditions must be met when using the binomial distribution.

1) There are a fixed number of trials, n.

2) There only two outcomes, "success" and "failure".

3) The trials are **independent**.

4) There is a constant probability of success p.

5) The random variable, X, is the total number of successes in n trials.

Common Examples

Common examples of experiments where the binomial distribution can be used to calculate probabilities are:

a) Flipping a coin 5 times and counting the number of heads we get.

b) Rolling a dice 8 times and counting the number of times we get an even number.

c) Picking a counter out of bag of 4 red and 3 green and getting a green 3 times when you conduct the experiment 6 times.

Let's look at the formula in more detail...

If $X \sim Bin(n, p)$

$$P(X = x) = \begin{cases} \binom{n}{x} p^x q^{n-x} & \text{for } x = 0, 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

where $0 < p < 1$ and $q = 1 - p$

Let's use it to calculate some probabilities...

A random experiment consists of tossing a fair coin three times. Let the random variable X represent the number of heads which appear. Let's check our conditions first to make sure we can use the binomial distribution:

1) There are a fixed number of trials n.

2) There only two outcomes, "success" and " failure".

3) The trials are **independent**.

4) There is a constant probability of success p.

5) The random variable, X, is the total number of successes in n trials.

Let's use it to calculate some probabilities...

A random experiment consists of tossing a fair coin three times. Let the random variable X represent the number of heads which appear. We use the following binomial model X ~ Bin (3, 0.5)

$$P(X = x) = \begin{cases} \binom{n}{x} p^x q^{n-x} & \text{for } x = 0, 1, 2, \ldots, n \\ 0 \text{ otherwise} & \end{cases}$$

where $0 < p < 1$ and $q = 1 - p$

Expectation and Variance

It can be shown that the for a binomial distribution with, n trials and a probability of success, p or in other words $X \sim Bin(n, p)$ that...

$E(X) = np$

$Var(X) = npq$

Calculating Probabilities

We have 3 ways to actually calculate the probabilities.

1) Using the formula.

2) Using the tables in the data booklet.

3) Using your calculator.

Example

Suppose $X \sim Bin(7, 0.4)$

Calculate

$P(X = 4) = 0.1935$

$P(X = 6) = 0.0172$

$P(X \leqslant 5) = 0.9812$

Continue to Practice using the Binomial Distribution

Make sure you have completed these questions, if not today then during a study period.

Ex 2.3A - Questions 5, 6, 7 and 8

Ex 2.3B - Questions 3, 4, 5 and 6

Starter

A researcher is studying woodland rodents as hosts for parasite transmission. The study involves capturing, examining, marking and releasing rodents on a number of sites in the Loch Lomond basin in the West of Scotland. The theoretical chance of a recapture (capturing a rodent that has previously been marked and released), determined from previous studies, is 20%.

(a) At one site the researcher captures 20 individuals. What is the probability that exactly 3 are recaptures?

(2017)

Poisson Distribution

Today we are learning...

What the Poisson distribution is and how we use it.

I know if I have been successful if...

I understand what the Poisson Distribution is.

I understand the conditions that must be met to use it.

I can calculate probabilities using it.

Poisson Distribution

Like the Binomial distribution the Poisson distribution is a discrete probability distribution. We can use it under the following conditions:

1) Isolated events are occurring in continuous time or space.

2) Events are independent.

3) Events occur singly rather than in groups.

4) Events are occurring at a constant average rate per unit of time through the period of interest.

5) The random variable X is the number of events occurring per time segment.

Examples of a Poisson Distribution

1) The number of telephone calls received at a particular switchboard during business hours with an average of 8 calls every 5 minutes.

2) The number of births in a hospital per hour.

3) The number of red cars passing a point on the road outside school in 3 minutes.

Poisson Distribution

If the random variable X is said to follow a Poisson Distribution then we write $X \sim Po(\lambda)$

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

The number of telephone calls received at a particular switchboard during business hours is thought to be a Poisson random variable with an average of 8 calls every 5 minutes.

$X \sim Po(8)$

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

Calculate...

$P(X = 4)$

$P(X = 10)$

$P(X \leqslant 13)$

EX2.4A - Page 48

EX2.4B - Page 49

Approximating the Binomial with the Poisson

Today we are learning...

How to approximate the binomial distribution with the poisson distribution.

I will know if I have been successful if...

I understand why we may use an approximation.

I understand how to find the value of lambda.

I can calculate probabilities using different methods.

Binomial Approximating the Poisson

Sometimes when we know a binomial distribution is the most appropriate model to help us calculate probabilities, to make calculation easier we sometimes instead approximate it using the Poisson distribution.

This may seem a little odd!

Example

A genetic disorder causes some people to have two different coloured eyes. This occurs in approximately 1 in 5 000 people.

If I select a random sample of 200 000 what is the probability 38 people have the genetic disorder?

The Relationship Between the Poisson and Binomial Distribution

$$\binom{n}{x} p^x q^{n-x} \approx \frac{e^{-\lambda}\lambda^x}{x!}$$

Only when $n \longrightarrow \infty$ and $p \longrightarrow 0$

As a general rule when applying this approximation n must be large (greater than or equal to 20) and p is small (less than or equal to 0.05) then...

$$X \sim Bin(n, p) \approx X \sim Poi(np)$$

Example

A genetic disorder causes some people to have two different coloured eyes. This occurs in approximately 1 in 5 000 people.

If I select a random sample of 200 000 what is the probability 38 people have the genetic disorder?

Let's look at the other example on Page 50 of the textbook...

EX2.5B - Page 50