Starter

Set up the **first** line of the solution for these percentage problems.

1) Lucy has a current account which earns 3·5% interest pa.
She deposits £3000 in the account and leaves it for 5 years.
How much money does she have at the end of the 5 years?

3000×1.035^5

2) Anna buys a boat for £20 000.
It depreciates in value by 5% every year.
What is it worth after 7 years?

$20\,000 \times 0.95^7$

3. Recurrence Relations

Recurrence Relations

Today's Learning:
By the end of this lesson, you should be able to use recurrence relations to solve problems.

A sequence is a list of numbers which follow a pattern.
We can define a list using a rule or formula.
Recurrence relations are one way of describing a sequence.

You have £2500 in a bank account which offers 4% interest pa.
How much do you have after 2 years?

National 5

Initial Amount: £2500

After one year: $1 \cdot 04 \times 2500 = £2600$

After two years: $1 \cdot 04 \times 2600 = £2704$

or

$1 \cdot 04^2 \times 2500 = £2704$

HIGHER

$u_0 = 2500$

$u_1 = 1 \cdot 04 \times u_0$

$u_2 = 1 \cdot 04 \times u_1$

$u_3 = 1 \cdot 04 \times u_2$

We can write this as: $u_{n+1} = 1 \cdot 04 u_n$ $\quad u_0 = 2500$

u_0 is the initial value.
u_n is the value after n years.
u_{n+1} is the value after $n+1$ years.

In general,

u_0	initial value
u_n	n^{th} term
u_{n+1}	the next term
$u_{n+1} = au_n$	recurrence relation

Example

$u_{n+1} = au_n$ recurrence relation

1) You invest £4500 in an account which offers 5·2% interest pa. Calculate the value in your account after 3 years.

$u_{n+1} = 1.052u_n$ $\quad u_0 = 4500$

$u_1 = 1.052 \times u_0$

$u_1 = 1.052 \times 4500 = 4734$

$u_2 = 1.052 \times u_1$

$u_2 = 1.052 \times 4734 = 4980.17$

$u_3 = 1.052 \times 4980.17 = £5239.14$

2) The value of an endowment policy increases at a rate of 6% per annum. If the initial value of the policy is £8000, find a recurrence relation for the value of the policy. Use this to calculate the value after four years.

$u_{n+1} = 1.06u_n$ $\quad u_0 = 8000$

$u_1 = 1.06 \times 8000 = 8480$

$u_2 = 1.06 \times 8480 = 8988.80$

$u_3 = 1.06 \times 8988.80 = 9528.128$

$u_4 = 1.06 \times 9528.128 = 10\,099.81568$

$\underline{£10\,099.82.}$

Linear Recurrence Relations

Today's Learning:
By the end of this lesson, you should be able to use more complex recurrence relations to solve problems.

Linear Recurrence Relations are of the form:

$$u_{n+1} = au_n + b \qquad a \neq 0, b \in \mathbb{R}$$

Example

1) A patient is injected with 250 ml of a drug. Every 6 hours, 28% of the drug passess out of the bloodstream. To compensate, a further 30 ml dose is given every 6 hours. Calculate the amount of drug remaining after 24 hours.

$u_{n+1} = 0.72u_n + 30$ $\quad u_0 = 250$

$u_1 = 0.72 \times 250 + 30 = 210$ 6

$u_2 = 0.72 \times 210 + 30 = 181.2$ 12

$u_3 = 0.72 \times 181.2 + 30 = 160.464$ 18

$u_4 = 0.72 \times 160.464 + 30 = \underline{\underline{145.53 \text{ ml}}}$ 24

2) A patient is injected with 160ml of a drug. Every 6 hours 25% of the drug passes out of her bloodstream. To compensate, a further 20ml dose is given every 6 hours.
a) Find a recurrence relation for the amount of drug in the bloodstream.
b) Use your answer to calculate the amount of drug remainging after 24 hours.

$u_0 = 160\text{ml}$ *75%* remains   *20ml* added

a) $u_{n+1} = 0{\cdot}75u_n + 20$

b) $u_1 = 0 \cdot 75 \times 160 + 20 = 140$ 6

$u_2 = 0 \cdot 75 \times 140 + 20 = 125$ 12

$u_3 = 0 \cdot 75 \times 125 + 20 = 113{\cdot}75$ 18

$u_4 = 0 \cdot 75 \times 113{\cdot}75 + 20 = 105{\cdot}3125$ 24

There is 105ml (to the nearest ml) left after 24 hours

2) A sequence is defined by the recurrence relation

$u_{n+1} = 0 \cdot 6u_n + 5 \qquad u_0 = 8$

Calculate the value of U_3 and find the smallest value of n, for which $u_n > 12$.

$u_{n+1} = 0 \cdot 6u_n + 5 \qquad u_0 = 8$

$u_1 = 0 \cdot 6 \times 8 + 5 = 9{\cdot}8$

$u_2 = 0 \cdot 6 \times 9{\cdot}8 + 5 = 10{\cdot}88$

$u_3 = 0 \cdot 6 \times 10{\cdot}88 + 5 = 11{\cdot}528$

$u_4 = 0 \cdot 6 \times 11{\cdot}528 + 5 = 11{\cdot}9168$

$u_5 = 0 \cdot 6 \times 11{\cdot}9168 + 5 = 12{\cdot}15008$

which is greater than 12

$u_3 = 0.6 \text{ x } 10\ 88 + 5 = 11{\cdot}528$ and $n = 5$ for $u_n > 12$

Starter

1) a) Thomas invests £4000 into a savings account which offers 5 ·4% interest per annum.
Each year he places £500 into the savings account.
Write this situation as a recurrence relation.

$u_{n+1} = 1 \cdot 054u_n + 500 \qquad u_0 = 4000$

b) How much will Thomas have in his account after 2 years?
You may use your calculator.

$u_2 = 1 \cdot 054u_1 + 500 = £5470 \cdot 66$

c) How long will it take Thomas to double his money?
You may use your calculator.

6 years

Does this recurrence relation have a limit?

$u_{n+1} = 0 \cdot 6u_n + 12 \qquad u_0 = 20$ (Use calc)

$u_8 = 29{\cdot}83$
$u_9 = 29{\cdot}898$
$u_{10} =$

Finding a Limit

Today's Learning:
By the end of this lesson, you should be able to recognise when a sequence will have a limit and find the limit.

$u_{n+1} = 0 \cdot 6u_n + 12$

Limit is 30 because

$0 \cdot 6 \times 30 + 12 = 30$ or

$30 = 0 \cdot 6 \times 30 + 12$

$L = 0 \cdot 6L + 12$
$-0{\cdot}6L \qquad -0{\cdot}6L$

$0 \cdot 4L = 12$

$L = \frac{12}{0 \cdot 4}$

$L = \frac{120}{4}$

$L = 30$

$u_{n+1} = 0{\cdot}5u_n + 2$
$L = 0{\cdot}5L + 2$

$$u_{n+1} = au_n + b$$

$$L = aL + b$$

$$L - aL = b$$

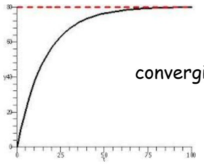
$$L(1 - a) = b$$

$$L = \frac{b}{1 - a}$$

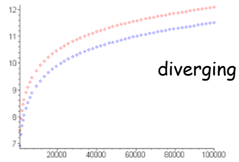
Convergence and Divergence

convergence means to tend towards a limit

divergence means not tending to a limit



converging



diverging

For linear recurrence relation $u_{n+1} = au_n + b$

the limit is defined as $L = \dfrac{b}{1 - a}$

For a sequence to have a limit $-1 < a < 1$

Example

1) The air pressure in a bouncy castle reduces by 12% from a day's use.
The owners increase the pressure by 9 units for the next day.
Initially the pressure was 70 units and it is dangerous to operate at greater than 74·5 units pressure.
Is this safe?

$u_{n+1} = au_n + b$

$u_{n+1} = 0 \cdot 88u_n + 9 \qquad u_0 = 70$

A limit exists as $-1 < 0 \cdot 88 < 1$

$$L = \frac{b}{1 - a}$$

$$= \frac{9}{1 - 0 \cdot 88}$$

$$= \frac{9}{0 \cdot 12}$$

$$= \frac{900}{12} \qquad = 75$$

This is not safe as the pressure will settle at around 75 units.

2) A pedestrian precinct has 80% of litter cleared each day by the council cleansing department.
However, 0·5kg is dropped each day.
The precinct is thought to be tidy if no more than 0·7kg of litter is scattered around.

Is it likely to appear tidy for the mayor's visit in six months time?

$U_{n+1} = 0 \cdot 2u_n + 0 \cdot 5$

limit exists as $-1 < 0 \cdot 2 < 1$

$L = \frac{b}{1-a}$

$= \frac{0 \cdot 5}{0 \cdot 8} = \frac{5}{8} = 0 \cdot 625$ kg

yes, it will appear tidy, $0 \cdot 625\text{kg} < 0 \cdot 7\text{kg}$

The two sequences defined by the recurrence relations $U_{n+1} = 0 \cdot 2U_n + 12$ (1) and $V_{n+1} = 0 \cdot 3V_n + k$ (2) have the same limit. The value of k is

A. 18

B. 4·5

C. 12

D. 10·5

$L = \frac{b}{1-a}$

(1) $L = \frac{12}{1-0 \cdot 2}$

$= \frac{12}{0 \cdot 8} = 15$

(2) $L = \frac{k}{1-0 \cdot 3}$

$= \frac{k}{0 \cdot 7}$

$15 = \frac{k}{0 \cdot 7}$

$10 \cdot 5 = k$

Note: $u_n = 0{\cdot}3\, u_{n-1} + 70$, $u_{n-1} = 1.12\, u_{n-2} + 25$, etc. are also recurrence relations.
The important thing is that there is a link between each term and the subsequent one.

$u_{n+1} = 0{\cdot}3u_n + 3$

$u_n = 0{\cdot}3u_{n-1} + 3$

$u_{n-1} = 0{\cdot}3u_{n-2} + 3$

Solving Recurrence Relations to find a and b

Today's Learning:
By the end of this lesson, you should be able to use two recurrence relations to find a and b.

If we know that a sequence is defined by a recurrence relation $u_{n+1} = au_n + b$ and we know several terms of the sequence, then we can find the values of a and b.

Examples

1) A sequence is defined by $u_{n+1} = au_n + b$ and $u_1 = 100$, $u_2 = 275$, $u_3 = 493{\cdot}75$

Find the values of a and b.

$u_2 = 100 \times a + b = 275$

$u_3 = 275 \times a + b = 493{\cdot}75$

$100a + b = 275$ ①
$275a + b = 493{\cdot}75$ ②

-1 × ① $-100a - b = -275$
② $275a + b = 493{\cdot}75$

Add $175a = 218{\cdot}75$
$a = 1{\cdot}25$

Sub $a = 1{\cdot}25$ into ① $100 \times 1{\cdot}25 + b = 275$
$125 + b = 275$
$b = 150$

The recurrence relation is defined by $u_{n+1} = 1{\cdot}25u_n + 150$

2) A sequence is defined by $u_{n+1} = au_n + b$ and $u_1 = 18{\cdot}5$, $u_2 = 20{\cdot}95$, $u_3 = 22{\cdot}665$

Find the values of a and b.

$u_2 = 18{\cdot}5 \times a + b = 20{\cdot}95$

$u_3 = 20{\cdot}95 \times a + b = 22{\cdot}665$

$18{\cdot}5a + b = 20{\cdot}95$ ①
$20{\cdot}95a + b = 22{\cdot}665$ ②

-1 × ① $-18{\cdot}5a - b = -20{\cdot}95$
② $20{\cdot}95a + b = 22{\cdot}665$

Add $2{\cdot}45a = 1{\cdot}715$
$a = 0{\cdot}7$

Sub $a = 0{\cdot}7$ into ① $18{\cdot}5 \times 0{\cdot}7 + b = 20{\cdot}95$
$12{\cdot}95 + b = 20{\cdot}95$
$b = 8$

The recurrence relation is defined by $u_{n+1} = 0{\cdot}7u_n + 8$

Attachments

maths1_4.ppt