CONTENTS

PREFACE

Mechanics is now a well established discipline which has vital relevance to science and technology. The fact that

- manned space flights take place
- high speed trains have been developed
- precision bombing is a reality

are all testament to the practical application of mechanical theories.

The subject itself can be divided into Statics (bodies at rest) and Dynamics (bodies in motion); the majority of the text is about describing motion and what causes motion. The final chapter, Physical Structures, is about Statics. Behind the application is the concept of a mathematical model, which represents mathematically the situation under study. This is usually, but not always, based on the assumptions of

Newton's Laws of Motion

These will be dealt with in this text, and their many applications developed. Above all it is our intention to show that mechanics is an interesting, practical and important topic, which embraces both the real world and our mathematical world.

This text has been produced for students and includes examples, activities and exercises. It should be noted that the activities are **not** optional but are an important part of the learning philosophy in which you are expected to take a very active part. The text integrates

- **Exposition** in which the concept is explained;
- **Examples** which show how the techniques are used;
- **Activities** which either introduce new concepts or reinforce techniques;
- **Discussion Points** which are essentially 'stop and think' points, where discussion with other students and teachers will be helpful;
- **Exercises** at the end of most sections in order to provide further practice;
- **Miscellaneous Exercises** at the end of most chapters, providing opportunities for reinforcement of the main points of the chapter.

Discussion points are written in a special typeface as illustrated here.

Note that answers to the exercises are given in a separate section. You are expected to have a calculator available throughout your study of this text and occasionally to have access to a computer.

Some of the sections, exercises and questions are marked with an asterisk (*). This means that they are either **not** central to the development of the topics in this text and can be omitted without causing problems, or they are regarded as particularly challenging.

Any enquiries regarding this text should be addressed to

Mathematics Enhancement Programme
CIMT, Faculty of Education
University of Plymouth
Douglas Avenue
Exmouth EX8 2AT

Tel: 01395 255521
Fax: 01395 255422

1 MODELLING and MECHANICS

Objectives

After studying this chapter you should

- understand what is meant by a mathematical model;
- appreciate what is meant by a force;
- understand how to model the force due to gravity.

1.0 Introduction

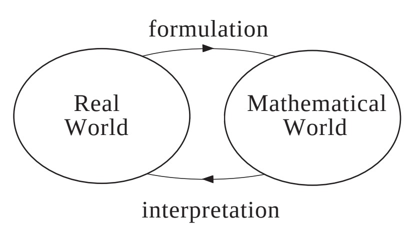
This chapter is about the use of mathematics in solving realistic problems. Traditionally the discipline in which the use of mathematics is studied has been called **Applied Mathematics**, and this term has often been associated with the application of mathematics to science and engineering. But mathematics occurs in many other subjects, for example in economics, biology, linguistics, transport as well as in industry, commerce and government. Applying mathematics to such a wide range of subjects requires not only good mathematical problem solving skills but also the ability of the mathematician to start with a problem in non-mathematical form and to give the results of any mathematical analysis in non-mathematical form. In between these start and end points, the mathematician must

formulate the problem into a form that allows the use of some mathematical analysis,

solve any mathematical problems that have been set up

and then

interpret the solution in the context of the original setting.



This process is called **mathematical modelling** and can be illustrated by the diagram opposite.

Central to mathematical modelling is the representation of the real world problem by a mathematical structure such as a graph, an equation or an inequality. Such a representation is what is meant by a **mathematical model**.

For example, the equation $s = 15t$ describes the distance travelled, s, in time t, when travelling at a speed of 15 ms^{-1}. The equation $s = 15t$ is a simple example of a mathematical model.

The first step in trying to devise a mathematical model for a given situation is to identify the quantities which can be measured and whose values will describe the real situation. In mechanics these quantities might be position, velocity, mass etc. In economics you might be interested in sums of money, inflation, depreciation, interest rates etc. Such quantities are called **variables**.

Broadly speaking, a mathematical model is a relation between two or more variables. The challenge to the applied mathematician is formulating a model which accurately describes or represents a given situation. To become skilful at mathematical modelling requires much hard work through experience gained at problem solving. This course on **mechanics** will provide the framework for you to learn about good models and to develop good problem solving skills.

1.1 Modelling from data

One of the simplest methods of finding a mathematical model for a given situation is to carry out an appropriate experiment to collect relevant data and from this data a formula relating the variables is found. This can be done using a graphic calculator, function graph plotter, spreadsheet or by simply drawing a graph. The graph, spreadsheet and the formula are mathematical models and can be used to describe the given situation and to predict what might happen for other variable values. This is a common method of mathematical modelling for scientific situations and most basic models in mechanics are found in this way.

In this section you will have the opportunity to apply this method of approach to several physical situations.

Activity 1 Finding models from experiments

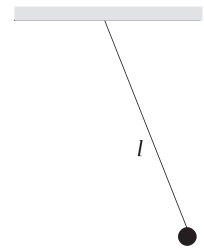
1. **Bouncing a ball**

   When a ball is dropped onto a table or hard floor it normally bounces back to a lower height above the table or floor. The cycle then repeats itself. Plan and carry out an experiment to investigate the relationship between the maximum height before each bounce and the maximum height after each bounce. Give details of any assumptions or simplifications that you have made.

2. **The period of a simple pendulum**

A simple pendulum consists of a small object attached to a fixed point by a string of length l. As the object swings back and forth the period of the pendulum is the time taken for one complete cycle.

Investigate the relationship between the period and the length of the pendulum. Does the result depend on the mass of the object?



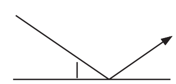
1.2 Motion and force

Motion

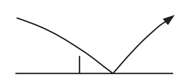
Mechanics is concerned with the motion of objects and what changes motion. The term **motion** means how objects move and how the way that they are moving is changing. In many cases motion is complicated; take for example the motion of a tennis ball from server to receiver. At first sight you might think that the ball moves in a straight line from the server's racquet to bounce to the receiver.

However, on closer observation the tennis ball moves in a curved path. In fact the motion of the ball could involve a swerve and/or a dipping in its path requiring a more complicated mathematical model. Such a model would involve the size and construction of the ball, the spin given to it and the effects of the air.

Instead of starting with such a complicated model you can simplify matters and consider first the translation of objects by ignoring rotations and spins, and follow the path of some **point** on or in the object. A description of the position, velocity and acceleration of this point then provides an answer to the question "what is motion?". You will begin to analyse motion mathematically in Chapter 2.



A very simple model



A curved path

Activity 2

1. For which of the following motions would the motion of some 'point' be sufficient? Where would you choose the 'point'?
   (a) a table tennis ball hit without spin;
   (b) a table tennis ball hit with spin;
   (c) a train on its journey from Plymouth to London;
   (d) a ball thrown horizontally from a tower;
   (e) an aircraft coming in to land at Heathrow;

(f) a ball rolling down a hill;

(g) a snooker ball sliding over the table (a 'stun shot');

(h) a swimmer diving from a springboard;

(i) cyclists going round a corner;

(j) a person in the chairoplane ride at Alton Towers.

2. In each of the ten situations above give a brief description of the motion.

Force

What is a force?

If an object which is initially at rest suddenly starts to move, then something must have caused the motion to start. For the object to start moving it must be acted upon by a net **force**. For example, a yacht moored in a harbour may begin to move when the sail is hoisted; the cause of the movement is the wind in the sail.

Suppose that you are towing a car up a 1-in-5 hill; it is important to know that the tow rope is strong enough not to break! What forces are acting in such a situation? You may have some idea already: there is the tension in the rope, the weight of the car, the reaction between the car and the road (hopefully the brakes are off!).

Examples of force are all around us. You probably have an intuitive idea of force from your own experience of pushing and pulling things. But what is a force? It is difficult to define a force precisely but what you can do is to describe the effect of a force. It has already been said that a force can start motion. A force is required to stop motion: for example, the American space shuttle returning from orbit glides towards the airforce base in California without any power. To stop the shuttle on landing, a force is produced by a parachute.

A force is required to make an object **move faster or slower**. For example an ice puck in ice hockey can glide across the ice in a straight line with constant speed; to make it go faster a player hits the puck. If the puck is hit in its direction of motion then it will continue in that direction. However, if hit across its path, the puck will **change its direction**.

So, in summary, a force can

- start motion;
- make an object move faster or slower;
- stop motion;
- change the direction of motion.

In other words a force **causes a change in motion**. As you will see, a force is **not** required to maintain motion. An object can move with constant speed with no force acting on it. For example a spacecraft in deep space, far from the influence of all planets, can maintain a constant speed without using its engines; only if it needs to change its motion does it need to uses its engines.

Activity 3

Consider the following situations and discuss whether there is a change in motion.

1. An apple falling off a tree.
2. A parachutist falling at a constant rate.
3. A bouncing ball while it is in contact with the ground.
4. A person in a car going round a roundabout.
5. An astronaut floating freely close to an orbiting satellite.

Types of force

There are essentially two types of force

- non-contact force;
- contact force.

The gravitational force between two objects is the best known **non-contact forc**e. It can usually be neglected unless one of the objects has a very large **mass**. In this chapter it is the gravitational attraction of the earth on an object that will be discussed, and this is usually called the **weight** of the object.

Imagine throwing a ball vertically upwards and watching it rise and then fall back to the ground. You can deduce that there must be a force acting on the ball which attracts it to the ground. This is the weight of the ball.

If you hold a golf ball in one hand and a shot (as in athletics) in the other hand you will feel the gravitational force of attraction even though neither object is moving. Furthermore, the effect on each hand will be different. The hand with the shot in it will ache long before the one with the golf ball in it! The **force of gravity** on an object depends on the mass of the object.

You will see later that the laws of motion stated by Newton lead to a law for the **weight** of an object. Furthermore the same law also defines the unit of force called the **Newton** (N).

> The weight of an object of mass 1 kg is approximately 9.8 Newtons

Often, for ease in calculations the weight of an object of mass 1 kg is taken as 10 N.

> The weight of an object of mass m kg is $9.8m$ Newtons

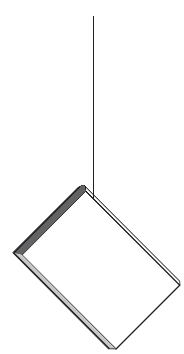
(The weight of an average sized apple is about 1 Newton!).

It is important to emphasize the difference between mass and weight, since the words are often interchanged and confused in everyday usage. The **mass** of a body is a measure of how much matter it contains and is also a measure of its reluctance to be accelerated by a force while the **weight** is the gravitational force exerted on it.

The mass of an object is independent of its position in the solar system. It is just a number associated with the object. However the weight of an object may depend on the object's position. For example, the weight of an athletics shot on the earth's surface is roughly 72 Newtons, but on the moon it would be only 12 Newtons. Thus on the moon you could hold six shots as easily as one on the earth. In each case however the mass of the shot is the same, roughly 7.2 kg. So when you are told that 'a bag of sugar weighs 1 kg' this is a concise way of saying that the weight is that of a 1 kg mass. To be mathematically correct you should say that 'the bag of sugar has a mass of 1 kg'.

All other forces you come across in this chapter are **contact forces** in which there must be contact between two surfaces for the force to exist. Words such as push, pull, tension, hit, knock, load, friction are used to describe contact forces.

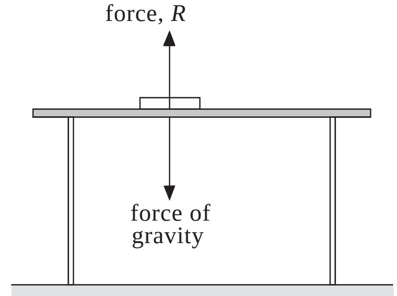
Tension force



Consider what happens when this book is suspended from the end of a string which is strong enough to support the weight of the book.

You know that gravity exerts a downward force of magnitude roughly 10 m Newtons on the book (where m is the mass of the book) and yet it remains at rest. This can only occur if there is an upward force which cancels out or balances the effect of gravity. This force on the book must be provided by the string. The pull that the string is exerting is called the **tension** in the string. Tension is the force with which a string, spring or cable **pulls** on what is attached to its ends.

Normal contact force and friction



Consider now this book lying on a horizontal table. The book is not moving, although there is the force of gravity acting on it. Take the table away and the book crashes to the floor. So when at rest on the table there must be at least one other force acting on the book which cancels out or balances the weight of the book.

This force is called the **normal contact force** (or sometimes the **normal reaction**).

The direction of the normal contact force is at right angles (or is 'normal') to the table. In this case the magnitude of the normal contact force is equal to the magnitude of the weight of the book. (Note that on a diagram a force is denoted by an arrow pointing in the direction of the force). Whenever two objects are in contact (in this case the objects are the table and the book) there is a contact force between them. The table exerts a force upwards on the book and the book exerts a force downwards on the table.

Activity 4

Suppose that you tip the table slowly so that the angle of the table top to the horizontal is small. Why does the book remain stationary? What could happen as the angle of the table top is gradually increased?

Exercise 1A

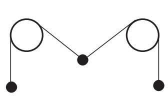
1. Two teams are involved in a tug of war contest. Draw a diagram of the situation using arrows to represent the forces that the rope exerts on each team.

   If the teams are equally balanced how do you think that these forces compare

   (a) in magnitude, and

   (b) in direction?

2. Consider the physical situation shown in the diagram below.

   The objects on each end of the string are identical having the same mass (0.1 kg) and shape. What forces act on each object? What is the magnitude of the tension in each string?

3. A child is hanging on to a rope tied to the branch of a tree. Draw diagrams to show:

   (a) the forces acting on the child;

   (b) the forces on the rope (assume it has zero mass);

   (c) the force that the rope exerts on the tree.

4. While running, an athlete has one foot in contact with the ground. Identify the forces acting

   (a) on the athlete;

   (b) on the ground.

1.3 Newton's law of gravitation

Earlier in this section the force of gravity on an object of mass m near the earth's surface was introduced as a force of magnitude $9.8m$ kg in the vertically downwards direction. Galileo, after dropping many objects from the leaning tower of Pisa, deduced that an object fell to the earth with an acceleration of $9.8\ \mathrm{ms}^{-2}$. So Newton with his second law was able to deduce the force on the object as $9.8m$.

However, one of Newton's great achievements was to generalize gravity in his **law of gravitation**. Newton postulated that the force of gravity on an object near the earth depended on the mass of the object, the mass of the earth and inversely on the square of the distance between the earth's centre and the object. To validate his law of gravitation, Newton clearly could not carry out experiments since he could not measure the mass of the earth and could not move objects very far from the earth's surface. Indeed, Newton studied the work of Kepler and realised that if he could 'prove' Kepler's laws using the law of gravitation and his law of motion, $F = ma$, then he was on to a winner! History tells us that Newton got it right and in the process developed Calculus.

Sir Isaac Newton (1642-1726) was, in fact, a prolific scientist and mathematician and turned his hand to a variety of problems including alchemy, optics and planetary motion. He was elected in 1689 as M.P. for Cambridge, and in 1699 accepted the post of Master of the Mint. Although he wrote that "if I have seen further than Descartes, it is because I have stood on the shoulders of giants", he undoubtedly must be regarded as one of the most influential mathematicians and scientists of all time.

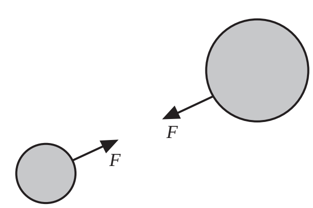
Throughout this book physical quantities will be measured in an agreed international system of units called SI units. In this system displacements are measured in metres (m) and time in seconds (s).

With the success of the law of gravitation applied to the planetary system, Newton postulated the **Universal Law of Gravitation.**

Every particle in the universe attracts every other particle in the universe with a force, F, that has magnitude (or size) directly proportional to the masses of the particles, m_1 and m_2, and inversely proportional to the square of their distance apart, d;

$$F = \frac{Gm_1m_2}{d^2}.$$

The proportionality constant G is called the gravitational constant and in SI units has the value $6.67\times10^{-11}\ \mathrm{m^3s^{-2}\ kg^{-1}}$.



Note that in formulating this law the word 'particle' has been used and in the introduction the importance of the point or particle model in mechanics was stressed. A particle is essentially a point in space of definite mass but no volume. For spherical objects, such as the earth or the moon, you can apply the law to a particle at the centre of the sphere with the mass of the sphere. For example, consider an object of mass *m* on the earth's surface. The radius of the earth is 6.37×10^{6} metres and its mass is 5.98×10^{24} kg. If you apply Newton's law of gravitiation to the object, then the force of attraction is

$$\begin{aligned} F &= \frac{\left(6.67\times10^{-11}\right)\times\left(5.98\times10^{24}\right)\times m}{\left(6.37\times10^{6}\right)^{2}} \\ &\approx 9.8\, m \text{ (in Newtons)} \end{aligned}$$

which agrees with the force of gravity law introduced at the beginning of this section. However, Newton’s law of gravitation provides a more general law which can be applied anywhere in the universe.

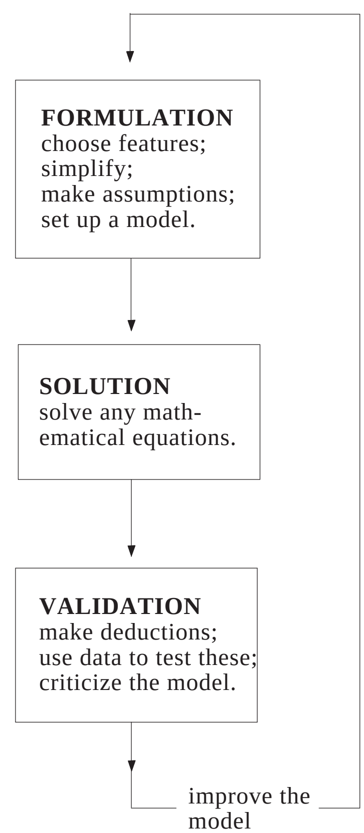
In the calculation for *F* the earth has been modelled as a particle of mass 5.98×10^{24} kg and the distance between the object of mass *m* and the 'particle earth' taken as 6.37×10^{6} metres.

The formulation of the law of gravitation demonstrates another form of mathematical modelling. In previous sections models have been formulated by doing experiments and collecting data. From this data the model in the form of a graph and/or an equation takes shape. This is called **empirical modelling**.

Newton’s work demonstrates a type of modelling called **theoretical modelling**. You can formulate a model based on an understanding of the physical situation and the important features that affect the situation. Newton proposed that the force between two objects depends on the masses of the objects and the distance between them. He then made three important **assumptions**:

- the force is an attractive force;
- the force is directly proportional to each mass;
- the force is inversely proportional to the square of the distance.

From these assumptions the model follows. Appropriate data or experimental activities are then used to validate the model. Notice how this approach uses the data at the end of the activity to test the model and not at the beginning to formulate the model. The diagram on the right of this page summarizes this type of modelling.



Exercise 1B

1. Determine the magnitude of the force between a man of mass 70 kg and a woman of mass 60 kg if they are 10 metres apart.
2. An astronaut of mass 75 kg is walking in space at a height of 3×10^7 metres measured from the centre of the earth. What force of gravity does the astronaut experience? How does this compare with the force of gravity on the earth's surface?
3. The mass of the moon is 7.38×10^{22} kg and its radius is 1.73×10^6 metres. What is the magnitude of the force of gravity on an object of mass m on the moon's surface?
4. Find the ratio of the force of gravity on an object on the earth's surface to the force of gravity on the moon's surface.
5. The distance between the centres of the earth and moon is 3.844×10^5 km. Determine how far from the centre of the earth an astronaut should be, so that the force of gravity of the earth exactly balances the force of gravity of the moon.

1.4 Units and dimensions

In mechanics, nearly all quantities are expressed in terms of units such as velocity in metres per second (ms^{-1}) and force in Newtons. The basic units in which quantities can be expressed are mass, length and time, denoted by the symbols M, L and T. For example, velocity can be expressed as

$$\frac{\text{L}}{\text{T}} \text{ or } \text{LT}^{-1}$$

and acceleration (ms^{-2}) as

$$\frac{\text{L}}{\text{T}^2} \text{ or } \text{LT}^{-2}.$$

The **dimensions** of velocity and acceleration are therefore LT^{-1} and LT^{-2} respectively. Since SI units are used for the measurements of quantities,

M is measured in kilograms,
L is measured in metres and
T is measured in seconds.

Some quantities such as force are not expressed in terms of the common SI units; however the dimensions can be established. For example, a farmer may be able to carry out a particular task at the rate of 5 hectares per hour. The dimensions of hectares, i.e. an area, are L^2, so

dimensions of hectares per hour are $\text{L}^2\ \text{T}^{-1}$.

The Universal Law of Gravitation, met in Section 1.3, is given as

$$F = \frac{G\, m_1 m_2}{d^2}$$

and the proportionality constant G, in SI units, has the value $6.67\times10^{-11} m^3 s^{-2} \mathrm{kg}^{-1}$ The dimensions of F are therefore given by combining the dimensions of G, m_1, m_2 and d^2.

So dimensions of $F = \dfrac{\dfrac{\mathrm{L}^3}{\mathrm{T}^2\mathrm{M}}\times\mathrm{M}\times\mathrm{M}}{\mathrm{L}^2}$

$$= \frac{\mathrm{L}^3\ \mathrm{M}}{\mathrm{T}^2\ \mathrm{L}^2}$$

$$= \mathrm{MLT}^{-2}.$$

Activity 5

Another way of expressing Force, F, is by using Newton's law of motion, $F = ma$, where m is the mass and a the acceleration.

Check that the dimensions given by this equation give the same dimensions as given above by the Universal Law of Gravitation.

Equations can be checked for dimensional consistency, in other words the dimensions on both sides of an equation should be the same.

Example

The equation for the period of oscillation (the time for one complete swing) of a simple pendulum is **thought** to be

$$P = \frac{2\pi}{g}\sqrt{l}$$

where P = period of oscillation (seconds)

g = acceleration due to gravity (ms^{-2})

l = length of pendulum (m).

Obviously the dimension of P is T.

The dimensions of the right hand side of the equation are

$$\frac{L^{\frac{1}{2}}}{LT^{-2}} = L^{-\frac{1}{2}}\ T^2.$$

The two sides of the equation are obviously not dimensionally consistent.

What changes to $\frac{2\pi}{g}\sqrt{l}$ are necessry so that the dimensions are consistent?

Experimenters sometimes use dimensions to evolve formulae. For example, it is known that the frequency (*f*) of a stretched wire depends on the mass per unit length, *m*, the length of the vibrating wire, *l*, and the force, *F*, used to stretch the wire. The problem is, therefore, to find the relationship between these quantities.

If you let

$$f = k\, m^a\ l^b\ F^c \quad \text{where } k = \text{constant},$$

then the dimensions of *f* must be the same as the dimensions of $m^a\ l^b\ F^c$.

The frequency means the number of vibrations per second, so its dimensions are T^{-1}.

So
$$T^{-1} = M^a\ L^{-a} \times L^b \times M^c\ L^c\ T^{-2c}$$
$$= M^{a+c} \times L^{-a+b+c} \times T^{-2c}$$

Equating the indices for *M L* and *T* on both sides of the equation gives

$$a + c = 0$$
$$-a + b + c = 0$$
$$-2c = -1$$

Hence $a = -\frac{1}{2}$, $b = -1$ and $c = \frac{1}{2}$.

So, the formula becomes

$$f = k\, m^{-\frac{1}{2}} l^{-1}\ F^{\frac{1}{2}}$$
$$f = \frac{k}{l}\sqrt{\frac{F}{m}}.$$

Exercise 1C

1. Which of the following equations are dimensionally consistent?

   (a) $t = \dfrac{v-u}{a}$ where t=time, v and u are velocities and a = acceleration.

   (b) $s = \dfrac{v^2-u^2}{2a}$ where s=distance, v and u are velocities and a = acceleration.

   (c) $t = \pi\sqrt{\dfrac{l^3}{g}}$ where t=period, l=length and g = acceleration.

   (d) $F = \dfrac{mu^2}{l} - mg(2-3\cos\theta)$

   where F=force, m=mass, u=velocity, l=length and g=acceleration.

2. Find the dimensions of each of these quantities which are commonly used in mechanics. Mass is represented by m, velocity by v, acceleration by g, length by h, force by F, time by t and area by A.

   (a) Kinetic energy E, given by $E = \frac{1}{2}mv^2$.

   (b) Potential energy P, given by $P = mgh$.

   (c) Impulse I, given by $I = Ft$

   (d) Pressure P, given by $P = \dfrac{F}{A}$

3. The frequency f of a note given by a wind instrument depends on the length l, the air pressure p and the air density d. If f is proportional to $l^x\ p^y\ d^z$ find the values of x, y and z.

4. The height h, at which a hover mower moves over the ground is thought to depend on the volume of air F which is pumped per second through the fan and on the speed u at which the air escapes from under the mower apron. Show that h, F and v could be related by the equation

   $h = k\sqrt{\dfrac{F}{v}}$.

2 ONE-DIMENSIONAL MOTION

Objectives

After studying this chapter you should

- be able to derive and use formulae involving constant acceleration;
- be able to understand the concept of force;
- be able to use Newton's Laws of Motion in various contexts;
- know how to formulate and solve equations of motion;
- be able to use the principles of conservation of momentum.

2.0 Introduction

The physical world is full of moving objects. **Kinematics** is the study of motion; **dynamics** is the study of forces that produce motion. In this chapter the mathematics for describing motion is developed and then the links between the forces acting and the change in motion are described.

To describe the motion of real objects you usually need to make simplifying assumptions. Perhaps the most important simplification in applied mathematics is ignoring the size and shape of an object. In Chapter 1 the notion of replacing a real object by a point was introduced. For example, in defining Newton's Law of Gravitation for the force acting on an object near to the earth's surface, the object and the earth were considered as points. Now the normal terminology is to consider objects as **particles**, and this is then called the **particle model**.

This simplification provides a starting point for many problems, but it does mean that some features of the motion of objects have to be ignored. For example, consider the description of the motion of a tennis ball. At first sight the ball may appear to follow the typical parabolic path of any object thrown near the earth's surface (such motions are studied in detail in Chapter 5). However, a closer study of the motion shows that the ball will be spinning, causing it to swing and dip. The particle model will be good enough to describe the overall parabolic motion but the effects of spin will have to be neglected. A less simple model which includes the features of size and shape would be required to describe the effects of spin.

Consider the motion of a snooker ball . It is possible to make the ball slide or roll on the table or to move with a combination of both types of motion. For which type of motion will the particle model be most appropriate? What features of the motion of a rolling snooker ball will be neglected with the particle model?

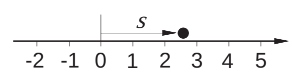
This chapter will concentrate on the description of objects which move along a **straight line**. The objects will be modelled as particles, and represented on diagrams by 'thick' points.

2.1 How to represent motion

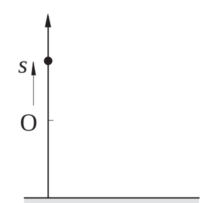
Displacement

Typical of the questions to be answered are: if a ball is thrown vertically upwards, how long does it take to fall to the floor? What is its velocity as it hits the floor?

To answer questions like these you need to find the **position** and **velocity** of the ball as functions of time. The first step is to represent the ball as a particle. The position of a particle moving in a straight line at any given instant of time is represented on a straight line by a single point. In order to describe the exact position you choose a directed axis with a fixed origin 0 and a scale as shown opposite.



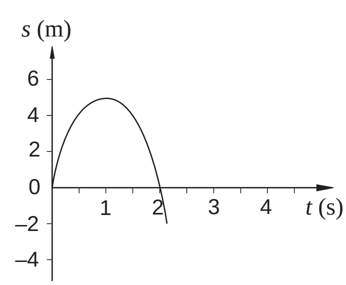
The position of the particle relative to the origin is called the **displacement** and is often denoted by the letter s measured in metres (m).



The choice of the origin in a problem will depend on what motion is being modelled. For example, in the problem of throwing a ball in the air an origin at the point of release would be a sensible choice.

Displacement - time graphs

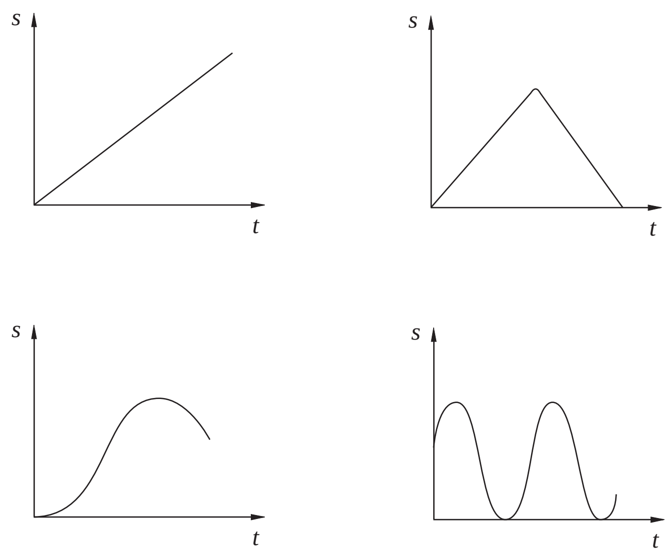
As a particle moves the displacement changes so that s is a function of time, t. A graphical method of showing motion is the **displacement-time graph** which is a plot of s against t. As an example, the figure shows the displacement-time graph for the motion of a ball thrown in the air and falling to the floor.



From the graph a qualitative description of how the position of the particle varies in time can be given. The ball starts at the origin and begins to move in the positive s direction (upwards) to a maximum height of 5 m above point of release. It then falls to the floor which is 2 m below the point of release. It takes just over 2 seconds to hit the floor but this part of the motion has not been shown.

Activity 1 Interpreting displacement - time graphs

Discuss the motion represented by each of the displacement - time graphs shown here.



Velocity

Once the position of a particle has been specified its motion can be described. But other quantities, such as its speed and acceleration, are often of interest. For example, when travelling along a road in a car it is not the position that is of interest to the police but the **speed** of the car!

The statement that the speed of a car on the M1 is 60 mph means that if the speed remains unchanged then the car travels for 60 miles in one hour. However the statement gives no information about the direction of motion. The statement that the **velocity** of a car on the M1 is 60 mph due north tells us two things about the car. First its speed is 60 mph (the **magnitude**) and the car is heading due north (the **direction**). Quantities which have magnitude and direction are called **vectors** and these are discussed more fully in Chapters 3 and 4.

The average velocity of a particle over a given time period, T say, is probably familiar to you and is defined by

$$\text{average velocity} = \frac{\text{displacement}}{\text{time taken}}.$$

Such a definition does not describe the many changes in velocity that may occur during the motion of the particle.

In the time interval T from $t = t_0$ to $t = t_0 + T$ the distance travelled is $s(t_0 + T) - s(t_0)$ so the average velocity is

$$\frac{s(t_0 + T) - s(t_0)}{T}$$

Now the quantity that is more interesting is not the average velocity of the particle but the **instantaneous** velocity. You will have seen from the definition of differentiation in the Foundation Core that the link between average changes and instantaneous changes is the derivative. As the time interval T tends to zero the ratio

$$\frac{s(t_0 + T) - s(t_0)}{T}$$

tends towards the derivative of $s(t)$.

Thus velocity is defined in the following way.

> If $s(t)$ is the displacement of a particle then its velocity is defined by
>
> $$v = \frac{ds}{dt}$$
>
> In the SI system of units velocity is measured in metres per second written as ms^{-1}.

Activity 2 Limits of average velocity

The displacement of a particle is given by

$$s = t^2 + 2t.$$

Calculate the average velocity of the particle during each of the time intervals

from $t = 1$ to $t = 2$

from $t = 1$ to $t = 1.1$

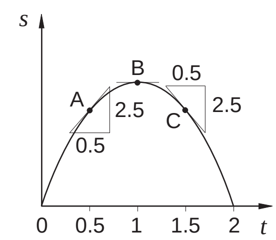
from $t = 1$ to $t = 1.01$

from $t = 1$ to $t = 1.001$.

Estimate the value that the average velocity is tending towards as $t \to 1$.

Does this value agree with $\dfrac{ds}{dt}$ when $t = 1$?

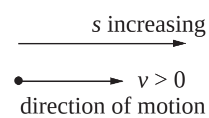
The definition of the velocity as a derivative can be interpreted geometrically as the slope of the tangent to the displacement - time graph. For example, consider the displacement - time graph opposite for the ball thrown into the air:



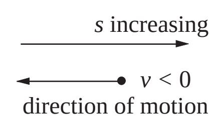
The slopes of the tangents to the graph at each of the points $t = 0.5$, $t = 1$ and $t = 1.5$ are equal to the velocities at these points.

When $t = 0.5$ (point A) the slope of the tangent is given by $\dfrac{2.5}{0.5} = 5\ \text{ms}^{-1}$. At $t = 1$ (point B) the slope is zero and at $t = 1.5$ (point C) the slope is $-5\ \text{ms}^{-1}$. You can now say more about the motion of the ball. At point A, the ball is going upwards with speed $5\ \text{ms}^{-1}$. At point B the ball is instantaneously at rest and it is at its highest point. At point C the ball is falling to the floor with speed $5\ \text{ms}^{-1}$.

For one-dimensional motion the sign of the velocity indicates the direction of motion of the particle.



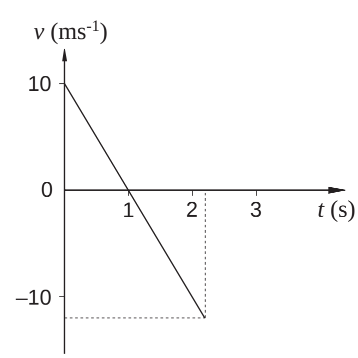
If $v > 0$ then s is increasing with time since $\dfrac{ds}{dt} > 0$.



If $v < 0$ then s is decreasing with time since $\dfrac{ds}{dt} < 0$.

The magnitude of the velocity of a particle is called its **speed**. For example, if $v = -3\ \text{ms}^{-1}$ then you can say that the particle moves with a speed $3\ \text{ms}^{-1}$ in the direction of s decreasing.

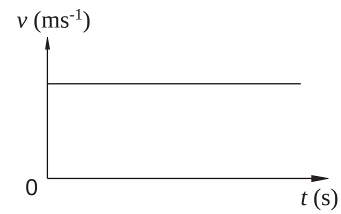
Velocity - time graphs



If you find the velocity, v, at several times, t, then a graph of the velocity against time is called a **velocity - time graph**. The figure shows the velocity - time graph for the motion of a ball thrown into the air and falling to the floor.

The ball begins its motion with a speed of $10\ \text{ms}^{-1}$ and this speed falls to zero during the first second of the motion. The speed then increases to approximately $12\ \text{ms}^{-1}$ when the ball hits the floor.

Activity 3 Interpreting velocity - time graphs



The diagram shows the velocity -time graph for Graph 1 in Activity 1.

Describe the motion of the particle from this graph.

Sketch the velocity - time graphs for the other displacement - time graphs in Activity 1.

Describe the motion of the particle in each case.

Acceleration

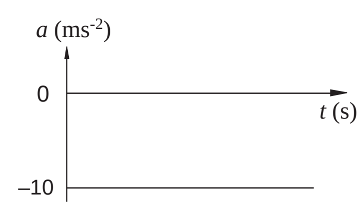
Chapter 1 identified the *change* in motion as an important quantity in the link between force and motion. Hence in many situations it is not the velocity that is important but the change in velocity. This is described by the **acceleration**.

You have seen that the velocity is defined as the rate of change of position; the acceleration is defined in a similar way to the velocity.

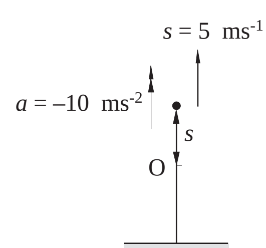
If $v(t)$ is the velocity of a particle at time t then the acceleration of the particle is defined by

$$a = \frac{dv}{dt}.$$

In the SI system of units, acceleration is measured in metres per second per second, written as ms^{-2}.



The acceleration can be obtained from the slope of the velocity - time graph. For example, for the motion of the ball thrown into the air, the diagram shows the acceleration - time graph. The acceleration is constant with magnitude 10 ms^{-2}. The negative sign implies that the velocity decreases continuously with time.



On a diagram velocities are shown with single arrows and accelerations with double arrows.

Activity 4 Interpreting acceleration - time graphs

Sketch the acceleration - time graphs from your velocity - time graphs in Activity 3.

Describe the motion of the particle in each case.

Relationships between displacement, velocity and acceleration

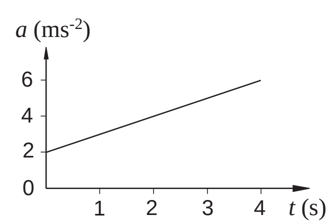
In this section you have seen that the motion of a particle along a straight line can be described by a displacement $s(t)$. The velocity and acceleration of the particle are then given by

$$v = \frac{ds}{dt} \quad \text{and} \quad a = \frac{dv}{dt}$$

respectively. Graphical descriptions of motion are given by displacement - time, velocity - time and acceleration - time graphs.

Now in mechanics problems the acceleration is often known and the velocity and displacement have to be found. This is achieved by integration. You will know from your knowledge of pure mathematics that integration is equivalent to finding the area under a graph.

Activity 5 Finding a velocity and displacement



The graph opposite shows the acceleration of a particle over 4 seconds of its motion. The particle starts from rest, at $t = 0$.

Estimate the velocity of the particle at the end of 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4 seconds.

Use your results to sketch a velocity - time graph for the particle.

From your velocity - time graph estimate the distance travelled during the 4 seconds of motion between $t = 0$ and $t = 4$.

The equation for the graph above is $a = 2 + t$. By integrating a to find v and then integrating v to find s, check your answer.

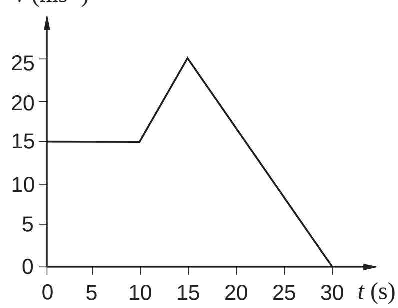
Exercise 2A

1. Sketch displacement - time and velocity - time graphs for the following:

   (a) A car starts from rest and increases its velocity steadily to 10 ms^{-1} in 5 seconds. The car holds this velocity for another 10 seconds and then slows steadily to rest in a further 10 seconds.

   (b) A ball is dropped on a horizontal floor from a height of 3 m. The ball bounces several times before coming to rest.

   (c) A person jumps out of an aircraft and falls until the parachute opens. The person glides steadily to the ground.

2. 

   The diagram represents the motion of an object for 30 seconds. Calculate the acceleration for each of the following intervals:

   (a) $0 < t < 10$ (b) $10 < t < 15$ (c) $15 < t < 30$

   Calculate the displacement of the object over the 30 seconds.

3.

t(s)	0	1	2	3	4	5	6	7	8
$v(\text{ms}^{-1})$	0	8	16	24	32	30	24	14	0

   (a) Plot these figures on a velocity - time graph.

   (b) Verify that for $0 \le t \le 4$, the figures are consistent with constant acceleration.

   (c) Calculate an estimate for the acceleration at $t = 6$.

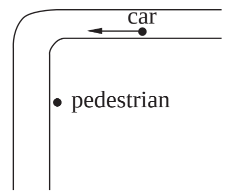
   (d) Calculate an estimate for the displacement after 8 seconds.

4. During the launch of a rocket the velocity was noted every second for 10 seconds and the following table of values obtained.

t(s)	0	1	2	3	4	5	6	7	8	9	10
v(kph)	0	32	80	128	176	224	272	320	368	400	448

   Estimate the distance travelled by the rocket during the first 10 seconds of its motion.

5. A road has a sharp bend around which approaching traffic cannot be seen. The width of the road is 6 m and the speed limit is 30 mph. The bend is on the near side of the road. How far from the bend should a pedestrian cross the road to avoid an accident? Assume the man walks at 3 mph.



6. A particle is set in motion at time $t = 0$ and its position is subsequently given by $s = 4 + 5t - 2t^2$.

   (a) Calculate the velocity of the particle after 1 second and after 2 seconds. What is the speed and direction of motion at each of these times?

   (b) Find the time at which the particle is instantaneously at rest.

   (c) Calculate the acceleration of the particle at $t = 1$ s and $t = 2$ s.

   (d) Describe the motion of the particle.

7. Repeat Problem 6 for

   (a) $s = t^2 - 4t + 1$

   (b) $s = 3 + 18t - 7.5t^2 + t^3$.

8. The acceleration of a particle is given by $a = -10\ \text{ms}^{-2}$. At the instant $t = 1$ s, the particle is at position $s = 2$ m and has velocity 3 ms^{-1}.

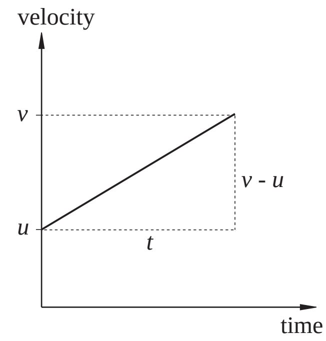
   (a) Find the velocity and displacement of the particle as functions of time.

   (b) Calculate the position and velocity of the particle when $t = 2$ s.

2.2 Modelling motion under constant acceleration

There are several simple formulae which can be used when dealing with motion under **constant** (also known as uniform) **acceleration**.

Discuss common types of motion where you think that constant acceleration is likely to occur.



The diagram represents the motion of an object with initial velocity *u* and final velocity *v* after *t* seconds has elapsed. The gradient of the line is calculated from the expression

$$\frac{v-u}{t}.$$

Since the gradient is equal to the value of the acceleration, *a*, then

$$a=\frac{v-u}{t}.$$

This can be rewritten as

$$\boxed{v=u+at} \qquad (1)$$

The area under the velocity - time graph is equal to the displacement of the object. Using the rule for the area of a trapezium gives

$$\boxed{s=\frac{(u+v)t}{2}} \qquad (2)$$

Note that $\frac{(u+v)}{2}$ is the average velocity of the object, so (2) is the algebraic form of the result that the displacement is equal to the average velocity multiplied by the time.

Example

A motorbike accelerates at a constant rate of 3 ms^{-2}. Calculate

(a) the time taken to accelerate from 20 mph to 40 mph.

(b) the distance in metres covered during this time.

Solution

You can use equation (1) to find the time and then equation (2) to find the distance travelled. But first you must convert mph to ms^{-1}.

Using the conversion rule 5 miles ≈ 8 kilometres, gives

$$10 \text{ mph} = 16 \text{ kph} = \frac{16 \times 1000}{60 \times 60} \text{ ms}^{-1}$$

$$= 4.445 \text{ ms}^{-1}.$$

Hence 20 mph $= 8.89 \text{ ms}^{-1}$ and 40 mph $= 17.78 \text{ ms}^{-1}$.

(a) Using equation (1) the time taken to accelerate from 8.89 ms^{-1} to 17.78 ms^{-1} at 3 ms^{-2} is given by

$$t = \frac{17.78 - 8.89}{3} = 2.96 \text{ s (to 3 sig. fig.)}.$$

(b) Using equation (2) the distance travelled in this time is

$$s = \left(\frac{8.89 + 17.78}{2}\right) \times 2.96 = 39.5 \text{ m}.$$

There are two further useful formulae which can be obtained from (1) and (2). Writing (1) in the form

$$t = \frac{v - u}{a}$$

and substituting into (2) gives

$$s = \left(\frac{u + v}{2}\right)\left(\frac{v - u}{a}\right),$$

so $$s = \left(\frac{v^2 - u^2}{2a}\right).$$ Remember

$v^2 - u^2 = (v - u)(v + u)$

This expression can be rearranged to give

$$v^2 = u^2 + 2as \qquad (3)$$

Similarly, substituting for v from (1) into (2) gives

$$s = \left(\frac{u + u + at}{2}\right) t$$

or $$s = ut + \tfrac{1}{2} at^2 \qquad (4)$$

Formula (3) is useful when the time t has not been given or is not required, while formula (4) is useful when the final velocity v has not been given nor is required.

Example

A car accelerates from a velocity of 16 ms^{-1} to a velocity of 40 ms^{-1} in a distance of 500 m. Calculate the acceleration of the car.

Solution

Using (3) $\quad 40^2 = 16^2 + 2 \times a \times 500$

$$a = \frac{1600 - 256}{1000} = 1.344 \text{ ms}^{-2}.$$

Example

A car decelerates from a velocity of 36 ms^{-1}. The magnitude of the deceleration is 3 ms^{-2}. Calculate the time required to cover a distance of 162 m.

Solution

When an object is slowing down it is said to be **decelerating**. You then use equations (1) - (4) but with a negative value for *a*. In this problem set $a = -3$.

Let *t* seconds be the time required to cover 162 m.

Using (4), $\quad 162 = 36t + \tfrac{1}{2} \times (-3) \times t^2$.

Rearranging this gives

$$1.5t^2 - 36t + 162 = 0.$$

Dividing by 1.5 gives

$$t^2 - 24t + 108 = 0.$$

Factorizing gives

$$(t-6)(t-18) = 0,$$

so that $\quad t = 6 \text{ or} 18.$

Discuss the significance of the two solutions to the quadratic equation. Which is the required time?

Activity 6 Stopping distance on the road

The Highway Code gives the following data for overall stopping distances of vehicles at various speeds.

Speed (mph)	Thinking Distance (metres)	Braking Distance (metres)	Stopping Distance (metres)
30	9	14	23
50	15	38	53
70	21	75	96

What is meant by the thinking distance?

Show that the deceleration during braking is roughly the same at each of the three speeds.

Use a graph plotter to fit a suitable curve through the data for speed and stopping distance. Use your results to estimate the speed corresponding to a stopping distance of 150 metres.

The amber warning light on traffic signals is intended to give drivers time to slow before a red stop light. Time the duration of amber lights in your locality. Do they give sufficient warning at the speed limit in operation on the road?

Exercise 2B

1. A car accelerates uniformly from a speed of 50 kph to a speed of 80 kph in 20 seconds.

   Calculate the acceleration in ms^{-2}.

2. For the car in Question 1, calculate the distance travelled during the 20 seconds.

3. A train signal is placed so that a train can decelerate uniformly from a speed of 96 kph to come to rest at the end of a platform. For passenger comfort the deceleration must be no greater than 0.4 ms^{-2}. Calculate

   (a) the shortest distance the signal can be from the platform;

   (b) the shortest time for the train to decelerate.

4. A rocket is travelling with a velocity of 80 ms^{-1}. The engines are switched on for 6 seconds and the rocket accelerates uniformly at 40 ms^{-2}. Calculate the distance travelled over the 6 seconds.

5. In 1987 the world record for the men's 60 m race was 6.41 seconds.

   (a) Assuming that the race was carried out under constant acceleration, calculate the acceleration of the runner and his speed at the end of the race.

   (b) Now assume that in a 100 m race the runner accelerates for the first 60 m and completes the race by running the next 40 m at the speed you calculated in (a).

   Calculate the time for the athlete to complete the race.

6. The world record for the men’s 100 m was 9.83 s in 1987. Assume that the last 40 m was run at constant speed and that the acceleration during the first 60 m was constant.

   (a) Calculate this speed.

   (b) Calculate the acceleration of the athlete.

7. Telegraph poles, 40 m apart stand alongside a railway line. The times taken for a locomotive to pass the two gaps between three consecutive poles are 2.5 seconds and 2.3 seconds respectively. Calculate the acceleration of the train and the speed past the first post.
8. The world record for the women's 60 m and 100 m are respectively 7.00 seconds and 10.49 seconds. Analyse this information using the method given in Question 6.
9. A set of traffic lights covers road repairs on one side of a road in a 30 mph speed limit area. The traffic lights are 80 m apart so time must be allowed to delay the light changing from green to red. Assuming that a car accelerates at $2\,\text{ms}^{-2}$ what is the least this time delay should be?
10. A van travelling at 40 mph skids to a halt in a distance of 15 m. Find the acceleration of the van and the time taken to stop, assuming that the deceleration is uniform.

2.3 How do bodies move under gravity?

For many centuries it was believed that:

(a) heavier bodies fell faster than light ones
and
(b) the speed of a falling body was constant.

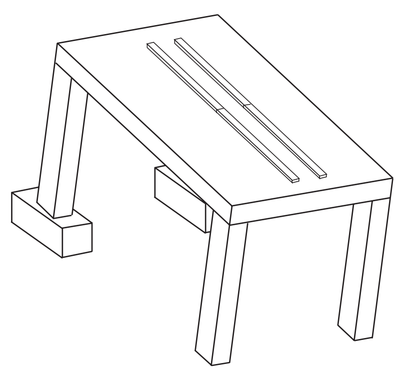
Discuss why such views were held and suggest ways in which they could be refuted.

Galileo Galilei (1564 - 1642) was the first person to state clearly that all objects fall with the same constant acceleration. Since there were no accurate timers available in the seventeenth century, he demonstrated this principle by timing balls which were allowed to roll down inclined planes.

Activity 7 Galileo's rolling ball experiment

You will need balls of different masses, a table, some blocks, four metre rulers and a stop watch.

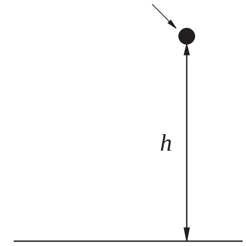
(a) Use the blocks to set up the table as shown in the diagram so that it takes about 4 seconds for the ball to roll down. Fix the rulers to make a channel down which the ball can roll. Measure the time it takes the ball to roll distances of 0.5, 0.75, 1, 1.25, 1.5, 1.75 and 2.0 m. Repeat the experiment and find the average time in each case.
(b) Find a relationship between the distance travelled, s metres, and time taken, t seconds.
(c) Do your results support Galileo's statement?



Modern determination of the acceleration of falling bodies gives values in the region of $9.81\ \text{ms}^{-2}$ although the value varies slightly over the surface of the Earth. The magnitude of this acceleration is denoted by g. A common approximation is to take $g = 10\ \text{ms}^{-2}$ and this value will be used in this text for solving problems.

Since the acceleration acts towards the Earth's surface, its sign must be opposite to that of any velocities which are upwards.

When dealing with problems involving motion under gravity, you can use the formulae for constant acceleration developed in the previous section.



Example

A ball is thrown vertically upwards with an initial speed of 30 ms^{-1}. Calculate the height reached.

Solution

Since the ball slows down, take $a = -g = -10 \text{ ms}^{-2}$.

Using (3) $\quad 0 = 30^2 - 2 \times 10 \times h$

where h is the maximum height reached.

Thus $\quad h = \dfrac{900}{20} = 45$ m (to 2 significant figures).

Example

A ball is thrown vertically upwards with a speed of 40 ms^{-1}. Calculate the time interval between the instants that the ball is 20 m above the point of release.

Solution

Using (4) with $a = -g = -10 \text{ ms}^{-2}$

$$20 = 40t - 0.5 \times 10 \times t^2$$

where t seconds is the time passed since the ball was thrown up.

$$5t^2 - 40t + 20 = 0$$

$$t^2 - 8t + 4 = 0.$$

Using the quadratic formula

$$t = \frac{8 \pm \sqrt{48}}{2} = 7.62 \text{ or } 0.54.$$

The ball is 20 m above the point of release twice, at $t = 0.54$ s (on way up) and $t = 7.62$ s (on way down).

The required time interval is $7.62 - 0.54 = 7.08$ seconds.

Two useful formulae which can be used on a body falling from rest through a height h metres can be found by putting $u = 0$, $a = 10$ and $s = h$ in equation (4) to give

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{h}{5}} \qquad (5)$$

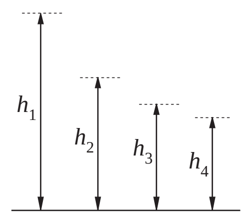
and by putting $u = 0$, $a = 10$ and $s = h$ in equation (3) to give

$$v = \sqrt{2gh} = \sqrt{20h} \qquad (6)$$

where t is the time of fall in seconds, v ms^{-1} is the final velocity and $g = 10$ ms^{-2}.

Activity 8 Estimating the value of g

You will need a bouncy ball and a metre rule for this activity.



Drop the ball onto a hard floor from a height of 2 m. Measure the height of the rebound. Repeat this several times and average your results. Now drop the ball from the rebound height and measure the new height of rebound. Repeat this procedure several times and average your results. Keep measuring new rebound heights for two further cases.

Now drop the ball from 2 m and measure the time elapsed until the fourth bounce with the floor. Repeat several times and average your results.

Show that the total time, t, up to the fourth bounce is

$$t = \sqrt{\frac{2h_1}{g}} + 2\sqrt{\frac{2h_2}{g}} + 2\sqrt{\frac{2h_3}{g}} + 2\sqrt{\frac{2h_4}{g}}.$$

Use this, together with your measurements, to calculate a value for g.

Exercise 2C

(Take $g = 10$ ms^{-2})

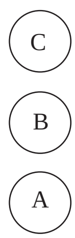
1. A ball is dropped on to level ground from a height of 20 m.

   (a) Calculate the time taken to reach the ground.

   The ball rebounds with half the speed it strikes the ground.

   (b) Calculate the time taken to reach the ground a second time.

2. A stone is thrown down from a high building with an initial velocity of 4 ms^{-1}. Calculate the time required for the stone to drop 30 m and its velocity at this time.
3. A ball is thrown vertically upwards from the top of a cliff which is 50 m high. The initial velocity of the ball is 25 ms^{-1}. Calculate the time taken to reach the bottom of the cliff and the velocity of the ball at that instant.
4. The diagram show three positions of a ball which has been thrown upwards with a velocity of u ms^{-1}



Position A is the initial position.

Position B is halfway up.

Position C is at the top of the motion.

Copy the diagram and for each position put on arrows where appropriate to show the direction of the velocity.

On the same diagram put on arrows to show the direction of the acceleration.

5. An aircraft is flying at a height of 4 km when it suddenly loses power and begins a vertical dive. The pilot can withstand a deceleration of 5 g before becoming unconscious. What is the lowest height that the pilot can pull out of the dive ?
6. If the Earth is assumed to be a perfect sphere then the acceleration due to gravity at a height h m above the surface of the Earth is given by

$$\frac{k}{(R+h)^2}$$

where R is the radius of the Earth in metres and k is a constant.

(a) Given that $R = 6400$ km and at the Earth's surface $g = 9.8\ ms^{-2}$, estimate the value of k.

(b) Use your calculator to find the height at which the acceleration differs by 1% from its value at the Earth’s surface.

(c) Use a graphic calculator to draw the variation of acceleration with height.

7. When a ball hits the ground it rebounds with half of the speed that it had when it hit the ground. If the ball is dropped from a height h, investigate the total distance travelled by the ball.
8. One stone is thrown upwards with a speed of 2 ms^{-1} and another is thrown downwards with a speed of 2 ms^{-1}. Both are thrown at the same time from a window 5 m above ground level.

(a) Which hits the ground first?

(b) Which is travelling fastest when it hits the ground?

(c) What is the total distance travelled by each stone?

2.4 What causes changes in motion?

Think about the following types of motion:

- an athlete running around a bend in a 200 m race;
- a ball being thrown over a fence;
- a car braking to a halt;
- a rocket accelerating in space;
- a snowball picking up snow as it rolls on level ground.

In each case there is a change in motion.

Discuss what these changes are in each case.

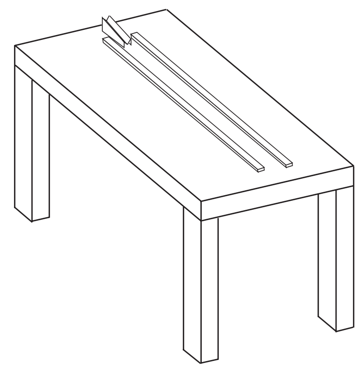
A change in motion is caused by **a force**. In medieval times it was thought that a force was required to keep a body in motion and that the only state which corresponded to an absence of forces was a state of rest.

The true relationship between forces and motion was stated by Newton using ideas of Galileo. In the absence of any forces there must be no change in the motion of the body, that is, the body must be at rest or moving with uniform velocity. Although first stated by Galileo, this is now generally known as Newton’s First Law of Motion.

Newton’s First Law

A body remains in a state of rest or moves with uniform motion, unless acted on by a force.

Activity 9   The chute experiment.



You will need a level table, 2 metre rules, a billiard ball, a chute and a stopwatch. You can make a chute out of a piece of folded cardboard.

Use the metre rules to make a channel on the table.

Allow the ball to roll down the chute so that it takes about 3 seconds to travel 1 m. Mark the point on the chute from which you release the ball.

Releasing the ball from this mark for each run, time the ball to roll 0.25, 0.5, 0.75 and 1 m.

Draw a displacement - time graph.

What forces act on the marble as it rolls on the table ?
Are your results consistent with Newton’s First Law?

If the total force acting on a body is **zero** (for example, if equal and opposite forces act on the same body), then the body remains at rest or moves in a state of uniform motion.

Newton’s First Law defines what happens to the motion of a body if no force is present. When a force acts on a body, there is a change in motion. Firstly, you need a clear definition of how to measure the motion of a body. Newton did a great deal of experimental work on this and came to the conclusion that the motion of a body was measured by the product of its mass and its velocity. This quantity is known as the **momentum** of the body.

$$\text{Momentum} = m \times v$$

A force produces a change in the momentum of a body, through a combination of changes in mass and/ or velocity.

Discuss the change in momentum in each of the cases given at the beginning of the section.

The physical law relating change in motion and the force acting on a body is given by Newton's Second Law.

Newton's Second Law

The rate of change of momentum of a body is proportional to the applied force.

In the case where the mass of the body is constant, this leads to the result that the force is proportional to the product of the mass and the acceleration of the body. By choosing appropriate units to measure mass, acceleration and force, the constant of proportionality can be made equal to one so that

$$\boxed{F = ma}$$

where m is in kilograms (kg)

a is in metres per second per second (ms^{-2})

F is in newtons.

One newton is the force sufficient to produce an acceleration of one ms^{-2} in a body of mass one kg. The abbreviation for a newton is N.

The approximate magnitude of some typical forces are

- force exerted by an adult arm ≈ 250 N
- force exerted by Earth's gravity on an adult ≈ 700 N
- force exerted by a car engine ≈ 2000 N

Example

A car of mass 1100 kg can accelerate from rest to a speed of 30 mph in 12 seconds. Calculate the force required.

Solution

Using the conversion $10 \text{ mph} = 4.445 \text{ ms}^{-1}$ gives

$$30 \text{ mph} = 13.3 \text{ ms}^{-1}$$

Assuming that the car accelerates at a constant rate, then using $v = u + at$ with $v = 13.3$, $u = 0$ and $t = 12$ gives

$$a = \frac{13.3}{12} = 1.11 \text{ ms}^{-2}.$$

Applying Newton's Second Law gives

$$F = ma$$

$$= 1100 \times 1.11$$

$$= 1220 \text{ N (to 3 significant places).}$$

Example

A force of magnitude 20 N is applied to a particle of mass 4 kg for 6 seconds. Given that the initial velocity of the body is 15 ms^{-1} ,

(a) calculate the acceleration, a, of the body;

(b) calculate its velocity, v, after 6 seconds.

After 6 seconds a force, F, is applied to bring the body to rest in a further 125 m.

(c) Calculate the magnitude of the force.

Solution

(a) Using Newton's Second Law,

$$20 = 4a$$

$$a = 5 \text{ ms}^{-2}$$

(b) Using (1), $v = 15 + 5 \times 6 = 45 \text{ ms}^{-1}$

(c) Since the particle is slowing down, the acceleration in equation (3) will be negative so put $a = -A$. Then using (3)

$$0 = 45^2 - 2 \times A \times 125.$$

Solving for A,

$$A = \frac{45^2}{250} = 8.1 \text{ ms}^{-2}$$

Using Newton's Second Law,

$$F = 4 \times 8.1 = 32.4 \text{ N.}$$

Activity 10 Forces produced by car engines

(a) Collect data on typical accelerations and masses of cars. Calculate the average force produced during the acceleration.

(b) Use the data for braking distances shown in Activity 6 to calculate the average force applied during braking.

Since all objects fall with an acceleration of g (neglecting air resistance) near the Earth's surface, the force acting on an object of mass m is given by

$$F = mg.$$

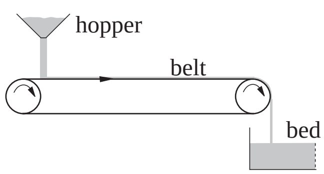
This force is known as the weight of the object.

Exercise 2D

1. A force of 50 N is applied to a particle of mass 4 kg for 5 seconds only.
   (a) Calculate the acceleration for the first 5 seconds.
   (b) Write down the acceleration for the next 5 seconds.
   (c) Calculate the distance travelled during the first 10 seconds given that the particle was at rest initially.

2. A world class sprinter can accelerate from rest to 10 ms^{-1} in about 2 seconds. Estimate the magnitude of the force required to produce this acceleration.

3. The brakes of a train are required to bring it to rest from a speed of 80 kph in a distance of 500 m. The mass of the train is 200 tonnes.
   (a) Calculate the average deceleration.
   (b) Calculate the average force required to be exerted by the brakes.

4. 

   The diagram shows a conveyor belt which is designed to convey coal dust from a hopper to a bed.

   Initially, there is no dust on the belt. The force required to drive the belt in this case is 200 N and the velocity of the belt is 2 ms^{-1}. The length of the belt is 40 m.

   Coal is now allowed to fall on the belt at a rate of 8 kg per second. The force driving the belt is adjusted so that the velocity stays at 2 ms^{-1}.

   Draw a diagram of force against time for the first 30 seconds of operation.

5. A toy car of mass 0.04 kg is propelled from rest by an engine which provides a pulling force of 2.7×10^{-3} N lasting for 8 seconds.
   (a) Calculate the acceleration.
   (b) Calculate the velocity after 8 seconds.

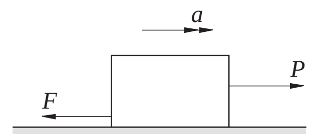
   If the speed of the toy car decreases uniformly to zero during the next 32 seconds, find the total distance travelled by the car.

6. A rocket has a mass of 40 tonnes of which 80% is fuel. The rocket motor develops a thrust of 1200 kN at all times.
   (a) Calculate the acceleration when the rocket is full of fuel.
   (b) Calculate the acceleration just before the fuel is exhausted.
   (c) What is the acceleration after the fuel is exhausted ?

7. A parachute reduces the speed of a parachutist of mass 70 kg from 40 ms^{-1} to 10 ms^{-1} in 3 seconds. Calculate the average force exerted by the parachute.

8. A car travelling at 30 mph is brought to rest in 3 seconds during a collision. Calculate the average force exerted on the car during the collision. Assume mass of driver is 70 kg and that of the car is 1100 kg.

2.5 How to deal with more than one force



The diagram shows a block of mass M kg being pulled by a force P N over a horizontal floor. The floor is rough so that there is a force, F N, due to frictional resistance acting in the opposite direction to P.

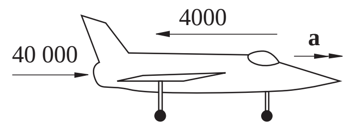
Since P and F act in opposite directions, their effect is a net force $P - F$ in the direction of P. Newton's Second Law is then applied using the net force to give

$$P - F = Ma \qquad (7)$$

where a is the acceleration of the block.

The net force is often called the **resultant** force.

Example



A jet aircraft of mass 8 tonnes has a single engine which generates a force of 40 000 N. Resistance to motion amounts to a constant value of 4000 N. [Reminder: 1 tonne $= 1000$ kg]

(a) Calculate the acceleration of the aircraft.

(b) The aircraft starts from rest. Calculate the speed after 12 seconds.

After 12 seconds the engine is switched off.

(c) Calculate the distance travelled by the aircraft before coming to a halt.

Solution

(a) The net force acting on the aircraft is $40\,000 - 4000$ N. So using equation (7), $40\,000 - 4000 = 8000a$.

Solving gives $a = 4.5 \text{ ms}^{-2}$.

(b) Equation (1) with $u = 0$, $a = 4.5$ and $t = 12$ gives

$$v = 0 + 4.5 \times 12 = 54 \text{ ms}^{-1}.$$

(c) The only horizontal force acting on the aircraft is resistance.

Using (7), $8000a = 0 - 4000$

$$\Rightarrow \quad a = -0.5 \text{ ms}^{-2}$$

where a is the acceleration of the aircraft when the engine is switched off. Since $a < 0$ the aircraft is slowing down.

Using (3), $0 = 54^2 + 2 \times (-0.5) \times s$

$$s = 2916 \text{ m}$$

where s is the distance travelled by the aircraft.

When equal but opposite forces act on a body, their net effect is zero, so there is no acceleration and the body remains at rest or in a state of constant velocity. Similarly if a body is at rest then the net force acting on the body must be zero. If the forces are not equal then there must be an acceleration.

In the diagram the sitter is not accelerating. Yet there is at least one force acting - his weight. Since people sitting on chairs do not accelerate downwards there must be an equal but opposite force acting. This force is provided by the chair acting on the sitter and is known as the normal contact force or reaction.

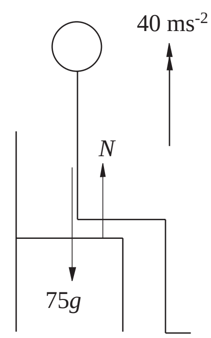
This leads to the following observation for two bodies in contact.

Newton's Third Law

Whatever the nature of the forces, for two bodies in contact, **equal** but **opposite** forces must act from one to the other.

Example

A manned rocket takes off with an acceleration of 40 ms^{-2}. An astronaut has a mass of 75 kg. Calculate the magnitude of the normal contact force acting from the chair on the astronaut.



Solution

The normal contact force is N. Using (7)

$$75 \times 40 = N - 75 \times g .$$

Using $g = 10$, the value of N is 3750 N.

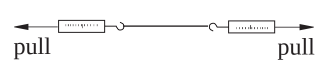
(This means that the astronaut will experience a force of 5 times her/his weight .)

Connected bodies

Bodies are generally pushed or pulled by strings, bars or chains. In a tug of war match, the opposing teams have no direct contact. If they are not accelerating then the pulling forces must be equal and opposite. Also each member is pulling so there must be an equal and opposite force in the rope. Forces in ropes and strings are known as tensions. Tensions act inwards along the ropes and act in equal but opposite pairs.

Activity 11 Tension and pulleys

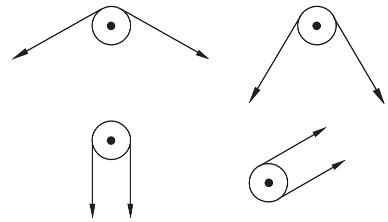
You will need a pulley, string and 2 newton-meters for this activity.



Tie each end of the string to a newton-meter as shown.

Pull to tension the string and note the readings.

What is the reading on one newton-meter if the other reads 2 N, 5 N, 9 N? Is it possible to get different readings?



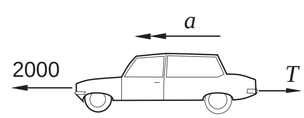
Now note the readings on the newton-meters if you tension the string over the pulley in each of the arrangements opposite.

Is it possible to get different readings on each newton-meter?

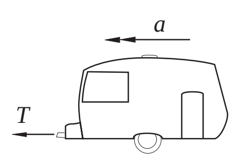
What can you say about the tension in the string on either side of a pulley?

Bars and rods can also support thrusts as well as tensions. Thrusts are denoted by outward facing arrows and they keep objects apart rather than bring them together. By using equal but opposite forces in connectors, it is possible to solve problems in which more than one body is involved.

Each body can be treated separately by considering the forces acting on it in isolation together with any tensions or thrusts from the connectors.



In the diagram a car of mass 1300 kg is connected to a caravan of mass 700 kg via a coupling. The car engine develops a pulling force of 2 kN. (1 kN = 1000 N)



The motion of the car and caravan can be considered separately. For the caravan, the pulling force is supplied by the tension in the coupling. The car is driven forward by the pulling force of the engine and held back by the tension, T, in the coupling.

For the car, using equation (7)

$$2000 - T = 1300a$$

where a is the acceleration of the car.

For the caravan

$$T = 700a$$

since the car and caravan must have the same acceleration.

Adding the two equations gives

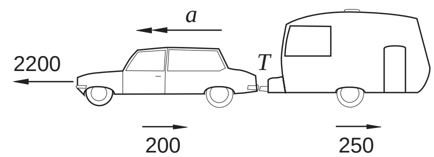
$$2000 = 2000a$$

$$\Rightarrow \quad a = 1 \text{ ms}^{-2}.$$

The tension in the coupling is $T = 700 \times 1 = 700 \text{ N}$.

Example

A car of mass 1100 kg pulls a caravan of mass 800 kg. The force exerted by the engine is 2200 N. In addition, friction and air resistance amount to 200 N on the car and 250 N on the caravan.



Calculate the acceleration of the car and the tension, T, in the coupling between the car and the caravan.

Solution

Using (7) for the car

$$2200 - 200 - T = 1100a$$

where a is the acceleration of the car.

Using (7) for the caravan

$$T - 250 = 800a.$$

Adding gives

$$1750 = 1900a$$

$$\Rightarrow \; a = 0.921 \text{ ms}^{-2}.$$

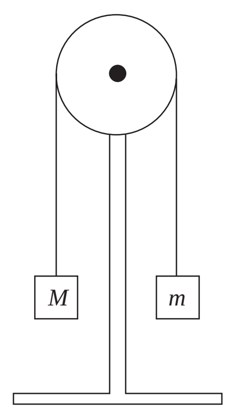
Now from $T - 250 = 800a$

$$T = 800 \times 0.921 + 250$$

$$= 987 \text{ N}.$$

Activity 12 Investigating the motion of connected bodies.

You will need a pulley, string, masses and a stop watch for this activity.



Set up the pulley and masses as shown in the diagram with the length of string slightly greater that the distance of the pulley from the floor. This distance should be about 1.5 m. Take $M = 120\text{g}$ and $m = 90\text{g}$.

Pull the 90 g mass down to the floor and measure the distance of the 120 g mass above it.

Release the system from rest and measure the time taken for the 120 g mass to reach the floor.

Repeat the measurements and average your results.

Use your results to calculate a value for the acceleration of the system assuming constant acceleration.

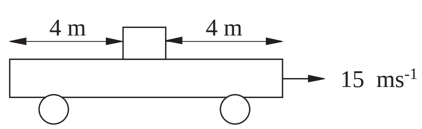
Derive an expression for the acceleration of the system in terms of M, m and g.

How well do your results agree? You might like to repeat the experiment for other sets of masses.

An assumption generally made in the motion of connected bodies is that the mass of the connector can be neglected compared to the masses of the bodies. Such connectors are called 'light'. Another assumption is that the length of the connector remains constant, that is, the connector does not stretch.

Discuss cases of motion where you think that one or both of the above assumptions would not be valid.

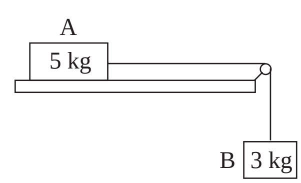
Exercise 2E

1. A block of wood of mass 500 gram is pulled by a force of 4 N. The resistance between the wood and the floor is 2 N. Calculate the acceleration.
2. Explain what you feel when you get into a lift which then ascends with an initial acceleration 2 ms^{-2} followed by a steady velocity and finally has a deceleration of 11 ms^{-2}.
3. During a car collision, the car's speed decreases from 50 mph to zero in 4 seconds. The driver is restrained by a car seat belt. Calculate the average force on the chest of an 80 kg driver.
4. 

   The diagram shows a box on the top of a trolley. The whole system is travelling on level ground at 15 ms^{-1}. The trolley reduces speed from 15 ms^{-1} to 5 ms^{-1} in 5 seconds. The mass of the box is 20 kg. As the trolley decelerates, the frictional force between the box and the trolley is 25 N.

   Describe, in detail, the motion of the box.
5. A shunting train pushes trucks along a level line. The mass of the locomotive is 20 tonnes and the mass of each truck is 4 tonnes. The locomotive can develop a force of 2000 N. Given that the locomotive pushes one truck and ignoring any resistances,

   (a) calculate the acceleration of the locomotive.

   (b) calculate the thrust on the truck.
6. 

   The diagram shows a mass A of 5 kg initially at rest on a horizontal table. A resistance force of 10 N acts against the motion of A which is connected to mass B of 3 kg by a light, inextensible string. The system is released from rest.

   (a) Calculate the acceleration of A.

   (b) Calculate the tension in the string.

   After a short time, B reaches the floor.

   (c) Calculate the acceleration of A now.
7. A breakdown truck tows a car of mass 1200 kg. Calculate the tension in the tow rope if the car is

   (a) accelerating at 0.5 ms^{-2} and experiencing a resistance force of 500 N;

   (b) travelling at constant speed but experiencing a resistance force of 400 N.

2.6 Forming differential equations

In Section 2.1 differentiation is used to describe the rates of change of displacement and velocity. The resulting equations, which include derivatives such as $\frac{ds}{dt}$ and $\frac{dv}{dt}$, are called **differential equations**.

A differential equation is a relation between a **function** and its **derivatives** with respect to some **variable**, often time or distance.

If a differential equation is used to model a situation, then a general method is needed to solve the differential equation.

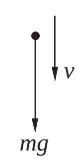
For a ball of mass m falling vertically downwards, neglecting air resistance, the equation of motion is given by Newton's Second Law

$$mg = m\frac{dv}{dt}$$

giving

$$\boxed{\frac{dv}{dt} = g} \qquad (8)$$

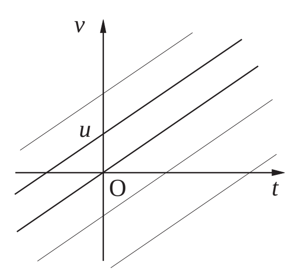
which is a first order differential equation with a constant right hand side. A **first order** differential equation is a relation between a function v and its first derivative $\frac{dv}{dt}$.



Integrating equation (8) with respect to t,

$$v = gt + C, \qquad (9)$$

where C is some constant. For different values of C the graphs of v against t are parallel straight lines with gradient g. To pick out a particular line it is necessary to specify v at some time t. If the ball is released from rest when $t = 0$, then



$$v = 0 \text{ when } t = 0.$$

Putting $t = 0$ in equation (9) gives

$$0 = 0 + C$$

so that $C = 0$ and equation (9) becomes

$$v = gt, \qquad (10)$$

which is the straight line passing through the origin O.

For the differential equation (8), expression (9) is called the **general solution**. The general solution of a first order differential equation has **one** arbitrary constant. The condition $v = 0$ when $t = 0$ is called an **initial condition** and expression (10) is a **particular solution**. A particular solution satisfies the differential equation **and** the initial condition.

If the ball had been given a velocity u at $t = 0$ then $v = u$ at $t = 0$ and, from equation (9),

$$u = 0 + C \quad \Rightarrow \quad C = u\,,$$

so that the particular solution is now

$$v = u + gt. \tag{11}$$

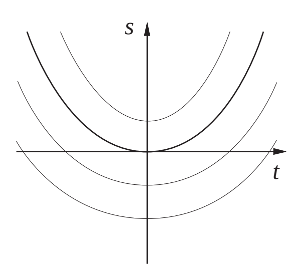
This straight line has intercept u on the v axis.

Since $v = \dfrac{ds}{dt}$, equation (11) gives

$$\frac{ds}{dt} = u + gt, \tag{12}$$

which is another first order differential equation.

Integrating with respect to t gives



$$s = ut + \frac{1}{2}gt^2 + B, \tag{13}$$

where B is some constant. Expression (13) is the general solution of equation (12). When $u = 0$, the graphs of s against t are parabolas with $t = 0$ as axis of symmetry. This family of parabolas have equal gradients

$$\frac{ds}{dt} = gt$$

at the same time t. If $s = 0$ when $t = 0$, then $B = 0$ and $s = \dfrac{1}{2}gt^2$.

Exercise 2F

1. Taking $g = 10\text{ms}^{-2}$ sketch the graphs of s against t from Equation (4),

   (a) if $s = 0$ when $t = 0$ and

   (i) $u = 5 \text{ ms}^{-1}$ (ii) $u = -10 \text{ ms}^{-1}$

   (b) if $s = 6$ m when $t = 0$ and $u = 10 \text{ ms}^{-1}$.

2. Integrate to find the general solutions of

   (a) $\frac{dy}{dt} = t^2$ (b) $\frac{dy}{dt} = \cos t$

   (c) $\frac{dy}{dt} = e^t$ (d) $\frac{dy}{dt} = -\frac{1}{t^2}$

   and sketch their graphs.

2.7 Direct integration

All the differential equations in the previous section are of the form

$$\frac{dy}{dt} = f(t), \tag{14}$$

where $f(t)$ is a function of t only. They can be solved by direct integration, giving

$$y = \int f(t)dt + C, \tag{15}$$

where C is some constant. Equation (15) is the general solution of equation (14).

So for example, if

$$\frac{dy}{dt} = t^3$$

then $$y = \int t^3 dt + C \quad \Rightarrow \quad y = \frac{1}{4}t^4 + C.$$

Example

Solve the differential equation

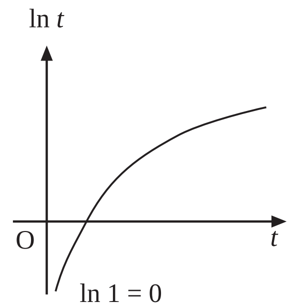
$$\frac{dy}{dt} = \frac{t+1}{t}$$

given that $y = 2$ when $t = 1$ for $t > 0$.

Solution

The general solution is

$$\begin{aligned} y &= \int \frac{t+1}{t}\,dt + C \\ &= \int \left(1+\frac{1}{t}\right)dt + C \\ &= \int dt + \int \frac{1}{t}\,dt + C \\ &= t + \ln t + C. \end{aligned}$$



When $t=1, y=2,$ so that

$$\begin{aligned} 2 &= 1 + \ln 1 + C \qquad (\ln 1 = 0) \\ &= 1 + C \end{aligned}$$

giving $C=1$ and the required solution is

$$y = t + \ln t + 1.$$

Exercise 2G

1. Find the general solutions of the following:

(a) $\dfrac{dy}{dt} = 2t^3 + 3$ (b) $\dfrac{dy}{dt} = \sin 2t$

(c) $\dfrac{dy}{dt} = \dfrac{1}{(1+3t)^{\frac{1}{2}}}$ (d) $\dfrac{dy}{dt} = (1+2t)^{\frac{1}{2}}$

(e) $\dfrac{dy}{dt} = \dfrac{1}{1+t}$ (f) $\dfrac{dy}{dt} = \dfrac{t}{1+2t^2}$.

2. Find the particular solutions of

(a) $\dfrac{dy}{dt} = 3t + 4, \quad y = 0$ when $t = 0,$

(b) $\dfrac{dy}{dt} = t^4 + \dfrac{1}{t^4}, \; y = 1$ when $t = 1,$

(c) $e^{2t}\dfrac{dy}{dt} + 1 = 0, \; y \to 2$ when $t \to \infty,$

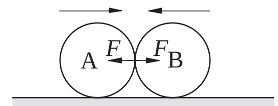
(d) $\dfrac{dy}{dt} = 2\sin 2t, \quad y = 1$ when $t = \dfrac{\pi}{4},$

and sketch their graphs.

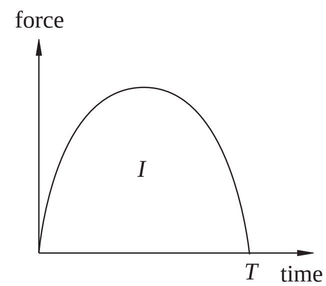
2.8 What happens during collisions?

When two particles collide, short term forces act during contact. The forces act to separate the particles although it is possible for them to stick together.

In the diagram opposite, A and B approach each other and have a head-on collision. During the collision, forces act to prevent the particles passing through each other. The force F acts from B to A for the time that they are in contact and an equal opposite one from A to B.



The area under the $F-t$ curve is known as the impulse of B on A. You can replace F by an average force F_{av} such that the area remains the same. In that case the impulse, I, is given by



$$I = F_{av}T$$

where T is the total time that the force acts.

Discuss whether it is true that being struck by a hard ball is more painful than being struck by a soft ball of the same mass travelling with the same speed.

What happens when you bring your fist down on a table - fleshy side first and then knuckles first?

Since there is a force acting from B to A, A will undergo a change of motion. Letting a be the average acceleration of A during the collision,

$$F = ma.$$

Now $$a = \frac{v-u}{T}$$

where u and v are the velocities of A before and after the collision respectively.

Combining the two equations gives

$$F = \frac{m(v-u)}{T} \quad \text{or}$$

$$FT = mv - mu. \qquad (16)$$

This is known as the **impulse equation** and is interpreted in the form

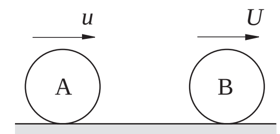
impulse on A = change in momentum of A.

Similar reasoning leads to

impulse on B = change in momentum of B.

When the two bodies are in contact, Newton's Third Law states that the impulses are equal and opposite.

Discuss why the impulses are equal and opposite.



Before impact

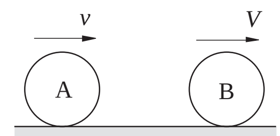
Adding the two equations above gives

change in momentum of A + change in momentum of B = 0.

This can be written algebraically to give

$$(mv - mu) + (MV - MU) = 0$$

or $$\boxed{MV + mv = MU + mu} \qquad (17)$$



After impact

In words, this can be expressed as

initial total momentum of the colliding bodies

= final total momentum of the colliding bodies.

This is known as the principle of **conservation of momentum**.

In the diagrams showing the motion before and after impact, the two bodies are shown as if they are always moving in the same direction, i.e. to the right. If, in fact, the bodies are moving towards each other as shown by the first diagram on the previous page, then the value of U must be negative when used in equation (17).

Example

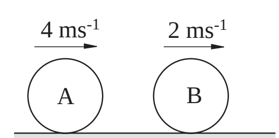
In the diagram A and B have masses 2 kg and 3 kg and move with initial velocities as shown. The collision reduces the velocity of A to 1 ms^{-1}. Find the velocity of B after the collision.

Solution

Impulse on A $= 2 \times 1 - 2 \times 4 = -6 \text{ kg ms}^{-1}$.

Impulse on B $= +6 \text{ kg ms}^{-1}$.

Using the impulse equation



$$+6 = 3v - 3 \times 2$$

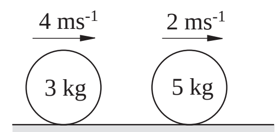
where v is the velocity of B after the collision, giving

$$v = 4 \text{ ms}^{-1}.$$

In general, if the masses and original velocities are known, then the algebraic form of the conservation of momentum is a single equation in the two unknowns V and v. So one of V and v must be measured in order to predict the other. There is one exception to this that is easy to deal with. This occurs when the bodies stick together or **coalesce**. In that case $V = v$.

Example

The two bodies in the diagram have masses 3 kg and 5 kg respectively. They are travelling with speeds 4 ms^{-1} and 2 ms^{-1}. The bodies coalesce on impact. Find the speed of this body.



Solution

Letting the final common velocity of the coalesced body be v, the principle of conservation of momentum gives

initial total momentum = final total momentum

$$\Rightarrow \quad 3\times4+5\times2=(3+5)\times v$$

$$\Rightarrow \quad v=\frac{22}{8}=2.75 \text{ ms}^{-1}.$$

Case study

Police forces have a well-established procedure for analysing car crashes. One of the simplest cases to consider is a head-on collision with a stationary vehicle.

Discuss the stages in the incident starting from the instant the car driver sees the stationary vehicle and the data which the police would have to collect to analyse the incident.

Example

A car with total mass, including passengers, of 1300 kg collided with a stationary car of mass 1100 kg parked without lights on a dark road. The police measured the length L metres of the skid marks of the moving car before impact and the length l metres of the skid tracks after impact. They also took note of the state of the road. In order to estimate the decelerations of the cars, the police drove a car under the same road conditions at 50 mph and found the length of the skid to be 30 m. L was found to be 14 and l to be 3. It is required to find the speed of the moving car before impact.

Solution

First of all, define symbols

U = speed of car before braking in ms^{-1}

V = speed of car just before impact in ms^{-1}

v = speed of the two cars just after impact in ms^{-1}

a = acceleration in ms^{-2} on the surface where the accident took place.

The acceleration is found from the trial skid.

Using (3)

$$0 = 22.2^2 + 2 \times a \times 30$$

$$a = -8.23 \text{ ms}^{-2}$$

The negative acceleration indicates deceleration, as expected.

Next, consider the motion after the collision

$$0 = v^2 - 2 \times 8.23 \times 3.$$

So $v = 7.03 \text{ ms}^{-1}$.

Conservation of momentum now yields

$$1300V = (1300 + 1100) \times 7.03$$

$$V = 13.0$$

The velocity of the car before braking can be found from (3).

$$13^2 = U^2 - 2 \times 8.23 \times 14$$

$$U = 20.0$$

So the initial speed of the car was 44.9 mph.

Exercise 2H

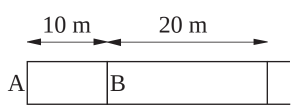
1. A particle A of mass 250 g collides with a particle B of mass 150 g. Initially A has velocity 7 ms^{-1} and B is at rest. After the collision, the velocity of B is 5 ms^{-1}.
   (a) Calculate the impulse of A on B.
   (b) Calculate the velocity of A after the impact.
2. Two railway trucks, each of mass 8 tonnes, are travelling in the same direction and along the same tracks with velocities 3 ms^{-1} and 1 ms^{-1} respectively. When the trucks collide they couple together. Calculate the velocity of the coupled trucks.
3. A service in tennis can result in a speed of 90 mph for the ball. The return of service is typically 60 mph. Given that the mass of a tennis ball is 60 g, calculate the impulse on the racket of the return of service.
4. A ball is dropped from a height of 2 m and rebounds to a height of 75 cm. Given that the mass of the ball is 70 g, calculate the magnitude and direction of the impulse of the floor on the ball.
5. The world record for the men's high jump is approximately 2 m 45 cm. Estimate the magnitude of the impulse needed for a 70 kg athlete to clear this height. [Hint: For a simple model assume that the athlete jumps vertically].
6. A child of mass 40 kg runs and jumps onto a skateboard of mass 4 kg. If the child was moving forward at 0.68 ms^{-1} when he jumped onto the skateboard, find the speed at which they move.
7. A tow truck of mass 3 tonnes is attached to a car of mass 1.2 tonnes by a rope. The truck is moving at a constant 3 ms^{-1} when the tow rope becomes taut and the car begins to move. Assume that both vehicles move at the same speed once the rope is taut, and find this speed.

2.9 Miscellaneous Exercises

1. Bill is going to take a penalty in a hockey game. He can hit the ball at a speed of 25 ms^{-1}. Initially the ball is placed 2.5 m from the goal line. The width of the goal is 1.5 m. John, the goalie, can move himself at a speed of 6 ms^{-1}. John stands in the middle of the goal.
   (a) What range of directions should Bill hit the ball to score? (Assume he hits the ball along the surface.)
   (b) What difference would it make to the answer to (a) if John's reaction time of 0.25 seconds were taken into consideration ?
   (c) Generalise to arbitrary velocities for John and Bill.

2. Two cars A and B are initially at rest side by side. A starts off on a straight track with an acceleration of 2 ms^{-2}. Five seconds later B starts off on a parallel track to A, with acceleration 3.125 ms^{-2}.
   (a) Calculate the distance travelled by A after 5 seconds.
   (b) Calculate the time taken for B to catch up A.
   (c) Find the speeds of A and B at that time.

3.



   The diagram shows part of an athletics track laid out for the changeover in a 4 by 100 m relay race. Assume the incoming runner A is moving at 10 ms^{-1} and the receiving runner B starts from rest with an acceleration of 4.5 ms^{-2}. The receiver starts running as soon as the incoming runner enters the box. Where does the changeover occur?

4. A motorist approaches a set of traffic lights at 45 mph. Her reaction time is 0.7 seconds and the maximum safe deceleration of the car is 6.5 ms^{-2}.
   (a) The motorist sees the lights turn to amber. What is the minimum distance from the lights that she can safely bring the car to a stop?
   (b) The lights remain on amber for 2 seconds, before turning red. As the motorist approaches the lights she sees them turn amber and decides to try to get past the lights before they turn red. What is the maximum distance from the lights that the motorist can do this at a constant speed of 45 mph?
   (c) Suggest an improvement to the model in (b), where constant speed was assumed. What difference does this make to the maximum distance ?
   (d) Generalise your answers to (a), (b) and (c) for an arbitrary approach velocity v ms^{-1}.

5. A college campus has a road passing through it. The speed of vehicles along the road is to be reduced by placing speed bumps at intervals along the road. The purpose of the bumps is to force vehicles to go very slowly over them.
   Suppose the maximum speed of a vehicle is 30 mph and the bumps are placed every D metres. Suppose the vehicle drives over a bump at 5 mph and its maximum acceleration is 2 ms^{-2}.
   (a) Sketch a velocity time diagram. Given that the vehicle just achieves its maximum speed, calculate the value of D, making simplifying assumptions where necessary.
   (b) Consider a range of speeds at which the vehicle crosses the bumps. Find the dependence of D on the speed, V, that the vehicle crosses the bumps. Assume a maximum speed of 20 mph.

6. A ball is thrown vertically upwards with an initial velocity of 30 ms^{-1}. One second later, another ball is thrown upwards with an initial velocity of u ms^{-1}. The particles collide after a further 2 seconds. Find the value of u.

7. The velocity of a car every 10 seconds is given in the following table.

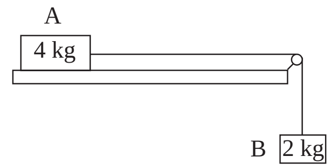
t (sec)	0	10	20	30	40	50	60
v(kph)	0	34	54	66	74	78	80

   Draw a velocity time graph for the motion of the car. Use your graph to estimate
   (a) the acceleration of the car after 25 seconds.
   (b) the displacement of the car after 60 seconds.

8. A force F N acts on a particle of mass 2 kg initially at rest. After 4 seconds the displacement of the particle is 20 m. Find the value of F.

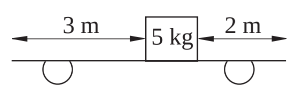
9. A locomotive has a pulling force of 125 kN and a mass of 120 tonnes. On a level track it travels at a steady speed of 72 kph.
   (a) What is the resistance to motion? Assuming that the resistance is proportional to the square of the velocity, find the constant of proportionality.
   (b) Calculate the acceleration at 54 kph.
10. A particle of mass 4 kg is released from rest and falls under gravity against a resistance to motion of $2v$ N where v is its velocity in ms^{-1}.

    Determine its terminal velocity. How long does it take to reach a velocity of 15 ms^{-1}?
11. A particle of mass 1 kg is released from rest and falls under gravity against resistance of $\frac{v^2}{2}$ N where v is its velocity in ms^{-1}. Determine its terminal velocity. How far does the body fall as its velocity increases from 2 ms^{-1} to 4 ms^{-1}?
12. Crowd control in some countries can involve the use of high pressure hoses. These spray water at people and can knock them over. The velocity of the water is 20 ms^{-1} and the jet has a diameter of 10 cm. Assuming that the momentum of the jet is destroyed on hitting the person, calculate the force on the person.
13. The gravitational attraction of the Earth on a particle of mass m is $\frac{km}{r^2}$ where r is the distance of the particle from the Earth's centre and k is a constant.

    Calculate the acceleration of a particle of mass m at a distance 120 km above the Earth's surface. [The radius of the Earth is 6400 km and at the Earth's surface $g = 9.8\ ms^{-2}$.]
14. A transport system is to be designed around a rail running over a set of pulse generators spaced 10 m apart and giving a force of 10 N acting 1 m either side of each generator to boost the velocity of the vehicle.
    (a) Sketch a velocity-time diagram.
    (b) Find the velocity increase after 55 m if the mass of the vehicle is 50 kg.
15. A locomotive of mass 80 tonnes pulls two trucks each with mass 9 tonnes. The pulling force of the locomotive is 14000 N. Resistance to motion can be ignored. Calculate the acceleration of the system and the tension in each of the couplings.
16. 

    Two bodies A and B of mass 4 kg and 2 kg respectively are attached by a light inextensible string passing over a smooth pulley. A rests on a table and B hangs over the side. Resistance forces on A amount to 8 N.

    The system is released from rest. Calculate the acceleration of the system and the tension in the string. Find the speed of B when it has fallen 2m.
17. A locomotive of mass M kg pulls a train of trucks. The mass of each truck is $\frac{M}{10}$ kg.

    The pulling force of the engine is F. How many trucks can the engine pull so that the tension in the coupling between the locomotive and the first truck is greater than $\frac{F}{2}$?
18. Rockets are made in sections known as stages. This is to enable a stage to be jettisoned once its fuel is used up. In a 3 stage rocket the mass of the lower stage is 1800 tonnes, the mass of the middle stage is 800 tonnes and the mass of the upper stage is 180 tonnes. The stages are coupled together by rings which can withstand a thrust of 1.65×10^7 N. Calculate the maximum safe acceleration of the rocket
    (a) when all 3 stages are present
    (b) when the upper 2 stages are present.
19. 

    The diagram shows a case of mass 5 kg on the floor of a trolley. Resistance force amounts to 5 N. The trolley is subject to a deceleration of 8 ms^{-2} until it is brought to rest from a speed of 40 ms^{-1}.

    Describe in detail the motion of the case.
20. A simple device for measuring acceleration (an accelerometer) consists of a mass m attached by a light inextensible string to the roof of a vehicle.

    Sketch diagrams to show what happens to this arrangement when :
    (a) the vehicle accelerates;
    (b) the vehicle decelerates;
    (c) the vehicle moves with uniform velocity.

21. A train of total mass 110 tonnes and velocity 80 kph crashes into a stationary locomotive of mass 70 tonnes.

(a) Calculate the velocity of the combined system immediately after impact.

The trains plough on for a further 40 m.

(b) Calculate the average deceleration and the resistance to motion.

22. A pile-driver consists of a pile of mass 200 kg and a driver of mass 40 kg. The driver drops on the pile with velocity 6 ms^{-1} and sticks to the top of the pile.

(a) Calculate the velocity of the pile immediately after impact.

Resistances to motion of the pile amount to 1400 N.

(b) Calculate the distance penetrated by the pile.

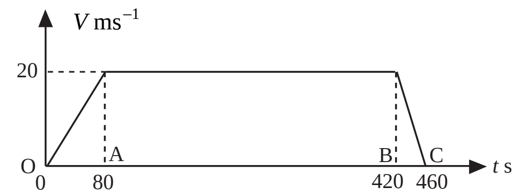
23. Two uniform smooth spheres, A of mass 0.03 kg and B of mass 0.1 kg, have equal radii and are moving directly towards each other with speeds of 7 ms^{-1} and 4 ms^{-1} respectively. The spheres collide directly and B is reduced to rest by the impact. State the magnitude of the impulse experienced by B and find the speed of A after impact. (AEB)

24. Two particles A and B of masses m and $2m$ respectively, are attached to the ends of a light inextensible string which passes over a fixed smooth pulley. The particles are released from rest with the parts of the string on each side of the pulley hanging vertically. When particle B has moved a distance h it receives an impulse which brings it momentarily to rest. Find, in terms of m, g and h, the magnitude of this impulse. (AEB)

25. A vehicle travelling on a straight horizontal track joining two points A and B accelerates at a constant rate of 0.25 ms^{-2} and decelerates at a constant rate of 1 ms^{-2}. It covers a distance of 2.0 km from A to B by accelerating from rest to a speed of v ms^{-1} and travelling at that speed until it starts to decelerate to rest. Express in terms of v the times taken for acceleration and deceleration.

Given that the total time for the journey is 2.5 minutes find a quadratic equation for v and determine v, explaining clearly the reason for your choice of the value of v. (AEB)

26.



The diagram shows the speed-time graph for a train which travels from rest in one station to rest at the next station. For each of the time intervals OA, AB and BC, state the value of the train's acceleration.

Calculate the distance between the stations. (AEB)

27. When a train accelerates its acceleration is always f km h^{-2} and when it decelerates its retardation is always $3f$ km h^{-2}. The acceleration is such that the train can accelerate from rest to 60 km h^{-1} in a distance of 1.5 km. Find

(a) f,

(b) the time taken to reach a speed of 60 km h^{-1} from rest,

(c) the distance travelled in decelerating from 60 km h^{-1} to rest.

On a journey of over 40 km the train is accelerated from rest to a speed of 60 km h^{-1} and kept at that speed until retarded to rest at the end of the journey. On one such journey the train is required, roughly half way through the journey, to slow down to rest, stay at rest for 3 minutes and then accelerate back to a speed of 60 km h^{-1}.

(d) Determine how late the train is on arrival at rest at its destination. (AEB)

28. Show that, in the usual notation, $v\dfrac{dv}{dx} = \dfrac{d^2x}{dt^2}$.

A particle P moves along the positive x-axis such that when its displacement from the origin O is x m, its acceleration in the positive x direction is $\left(10x - 2x^3\right)$ ms^{-2}. The speed of P is $\sqrt{15}$ ms^{-1} when $x = 2$. Find an expression for the speed of P for any value of x.

Determine the values of x for which P comes instantaneously to rest. (AEB)

29. A particle moving in a straight line with speed u ms^{-1} is retarded uniformly for 16 seconds so that its speed is reduced to $\frac{1}{4}u$ ms^{-1}. It travels at this reduced constant speed for a further 16 seconds. The particle is then brought to rest by applying a constant retardation for a further 8 seconds. Draw a time-speed graph and hence, or otherwise,

    (a) express both retardations in terms of u,

    (b) show that the total distance travelled over the two periods of retardation is $11u$ m,

    (c) find u given that the total distance travelled in the 40 seconds in which the speed is reduced from u ms^{-1} to zero is 45 m.

    (AEB)

30. A tram travelling along a straight track starts from rest and accelerates uniformly for 15 seconds. During this time it travels 135 metres. The tram now maintains a constant speed for a further one minute. It is finally brought to rest decelerating uniformly over a distance of 90 metres. Calculate the tram's acceleration and deceleration during the first and last stages of the journey. Also find the time taken and the distance travelled for the whole journey.

    (AEB)

31. A train travelling at 50 ms^{-1} applies its brakes on passing a yellow signal at a point A and decelerates uniformly, with a deceleration of 1 ms^{-2}, until it reaches a speed of 10 ms^{-1}. The train then travels for 2 km at the uniform speed of 10 ms^{-1} before passing a green signal. On passing the green signal the train accelerates uniformly, with acceleration 0.2 ms^{-2}, until it finally reaches a speed of 50 ms^{-1} at a point B. Find the distance AB and the time taken to travel that distance.

    (AEB)

3 VECTORS 1

Objectives

After studying this chapter you should

- understand that a vector has both magnitude and direction and be able to distinguish between vector and scalar quantities;
- understand and use the basic properties of vectors in the context of position, velocity and acceleration;
- be able to manipulate vectors in component form;
- recognise that vectors can be used in one, two and three dimensions;
- understand the significance of differentiation of vectors;
- be able to differentiate simple vector functions of time.

3.0 Introduction

> "Set course for Zeeton Mr Sulu, warp factor 5"
>
> "Bandits at 3 o'clock, 1000 yards and closing"

There are many situations in which simply to give the size of a quantity without its direction, or direction without size would be hopelessly inadequate. In the first statement above, both direction and speed are specified, in the second, both direction and distance. Another example in a different context is a snooker shot. Both strength and direction are vital to the success of the shot.

Activity 1 Size and direction

Suggest some more situations where both the size and direction of a quantity are important. For two of the situations write down why they are important.

Quantities which require size (often called magnitude) and direction to be specified are called **vector quantities**. They are very different from **scalar quantities** such as time or area, which are completely specified by their magnitude, a number.

Activity 2 Vectors or scalars?

Classify the following as either vector or scalar quantities: temperature, velocity, mass, length, displacement, force, speed, acceleration, volume.

'And after re-entry to earth's atmosphere, Challenger's velocity has been reduced to 800 mph.'

Discuss the statement above and whether the correct meaning is given to the terms speed, velocity and acceleration in everyday language.

Displacement

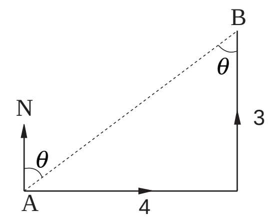
One of the most common vector quantities is **displacement**, that is distance and direction of an object from a fixed point.

Example

An aircraft takes off from an airport, A. After flying 4 miles east, it swings round to fly north. When it has flown 3 miles north to B, what is its displacement (distance and direction) from A?

Solution

The distance of from A is AB in the diagram shown opposite. Using Pythagoras' theorem



$$AB^2 = 4^2 + 3^2$$

$$\Rightarrow \quad AB = 5 \text{ miles.}$$

The direction of B from A is the bearing $\theta°$ where

$$\tan\theta = \frac{4}{3}$$

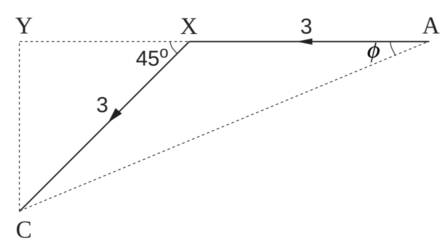
$$\Rightarrow \quad \theta = 53°.$$

The displacement of B from A is 5 miles on a bearing of $053°$.

Example

An aircraft takes off from A facing west and flies for 3 miles before swinging round to fly in a south-westerly direction to C. After it has flown for a further 3 miles, what is its displacement from A?

Solution

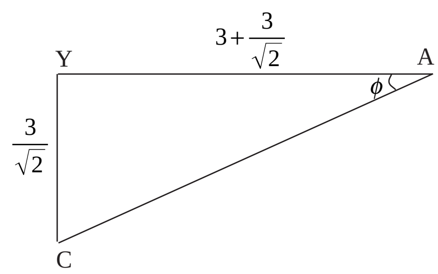


From Δ XYC

$$XY = 3\cos 45 = 2.12$$

$$YC = 3\cos 45 = 2.12.$$

Using Pythagoras' theorem on triangle AYC



$$AC^2 = (3+2.12)^2 + (2.12)^2$$

$$\Rightarrow \quad AC^2 = 26.23 + 4.50 = 30.73$$

$$\Rightarrow \quad AC = 5.54.$$

The direction is the bearing $(270-\phi)°$

where $\tan\phi = \frac{YC}{AY} = \frac{2.12}{5.12} = 0.414$

and so

$$\phi = 22.5°$$

The displacement of C from A is 5.54 miles on a bearing of $247.5°$.

Column vectors

Distance and bearing is only one method of describing the displacement of an aircraft from A. An alternative would be to use a **column vector**. In the first example above, the displacement of aircraft B from A would be $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$. This means the aircraft is 4 miles east and 3 miles north of A.

In the second example, the displacement of C from A is

$$\begin{pmatrix} -3-XY \\ -YC \end{pmatrix}$$

$$= \begin{pmatrix} -3-2.12 \\ -1.12 \end{pmatrix}$$

$$= \begin{pmatrix} -5.1 \\ -2.1 \end{pmatrix}.$$

Exercise 3A

Find the displacement of each of the following aircraft from A after flying the 2 legs given for a journey. For each, write the displacement using

(a) distance and bearing;

(b) column vector.

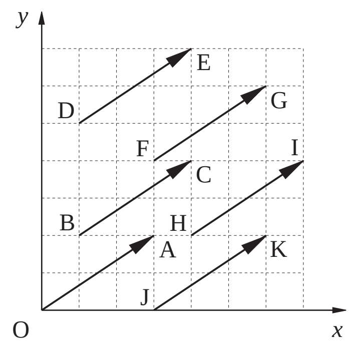
1. The aircraft flies 5 miles north then 12 miles east.
2. The aircraft flies 3 miles west then 6 miles north.
3. The aircraft flies 2 miles east then 5 miles south-east.

An aircraft is at a point X whose displacement from A is $\begin{pmatrix}-5\\6\end{pmatrix}$.

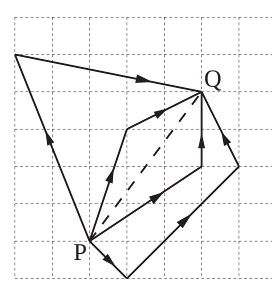
If there is no restriction on the direction it can fly, is there any way of knowing the route it took from A to X?

How is the column vector giving the displacement of A from X related to the column vector giving the displacement of X from A?

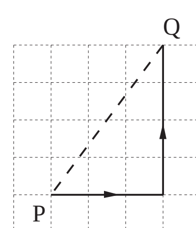
3.1 Vector notation and properties



In the diagram opposite, the displacement of A $(3, 2)$ from O $(0, 0)$ is described by the vector $\begin{pmatrix}3\\2\end{pmatrix}$. This time, the entries in the vector give distances in the positive *x* and positive *y* directions. The displacement of C $(4, 4)$ from B $(1, 2)$ is also $\begin{pmatrix}3\\2\end{pmatrix}$. In fact each of the line segments $\overrightarrow{OA}$, $\overrightarrow{BC}$, $\overrightarrow{DE}$, …, $\overrightarrow{JK}$ is represented by the vector $\begin{pmatrix}3\\2\end{pmatrix}$. In general, any displacement of '3 along, 2 up' has vector $\begin{pmatrix}3\\2\end{pmatrix}$. However $\overrightarrow{OA}$ is special. It is the only vector $\begin{pmatrix}3\\2\end{pmatrix}$ which starts at the origin. The **position vector** of the point A $(3, 2)$ relative to the origin 0 is said to be $\begin{pmatrix}3\\2\end{pmatrix}$.



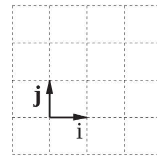
Any displacement such as $\overrightarrow{PQ}$ in the diagram can be thought of as the result of two or more separate displacements. Some possibilities are shown in the diagram, each starting at P and ending at Q.



Of the two stage displacements which are equivalent to the vector $\overrightarrow{PQ}$, only one has its first segment parallel to the positive *x* direction, and second parallel to the positive *y* direction. Because this is unique and is extremely useful, it has its own representation.

Components of a vector

A displacement of one unit in the positive x direction is labelled **i** and a displacement of one unit in the positive y direction is labelled **j**. Because each has length one unit, they are called **unit vectors**.



So $\qquad \overrightarrow{PQ} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

can also be written as

$$\overrightarrow{PQ} = 3\mathbf{i} + 4\mathbf{j}$$

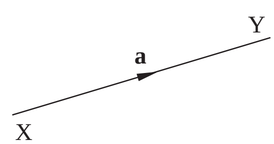
$3\mathbf{i}$ and $4\mathbf{j}$ are known as the **components** of the vector $\overrightarrow{PQ}$.

When working in three dimensions, a third unit vector **k** is introduced (see Section 3.4).

Notation

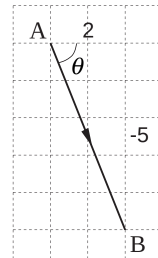
Besides using the end points of the line segment with an arrow above to denote a vector, you may see a single letter with a line underneath in handwritten text (e.g. $\underline{a}$) or the letter in bold type (e.g. **a)** without the line underneath in printed text.

In the diagram opposite, $\overrightarrow{XY}$ and $\mathbf{a}$ are two ways of referring to the vector shown. It should be noted also that $\overrightarrow{YX} = -\mathbf{a}$ is a vector of equal length but in the opposite direction to $\overrightarrow{XY}$ or $\mathbf{a}$.



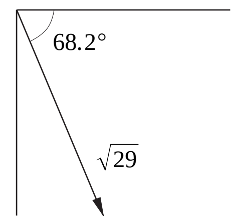
The reason why underlining letters has become the standard method of denoting a vector is because this is the instruction for a printer to print it in bold type. It is essential that you always remember to underline vectors, otherwise whoever reads your work will not know when you are using vectors or scalars.

Magnitude and direction of a vector



Consider the vector $\overrightarrow{AB} = 2\mathbf{i} - 5\mathbf{j}$ The magnitude or modulus of vector $\overrightarrow{AB}$, written $|\overrightarrow{AB}|$, is represented geometrically by the length of the line AB.

Using Pythagoras' theorem

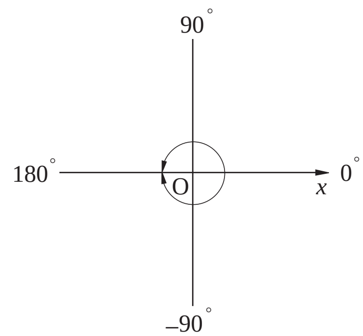


$$\begin{aligned}|\overrightarrow{AB}| &= \sqrt{2^2 + (-5)^2} \\ &= \sqrt{4+25} \\ &= \sqrt{29} \\ &= 5.39 \text{ units.}\end{aligned}$$

Its direction is defined by the angle AB makes with the positive x direction. This angle is $-\theta$, where

$$\tan\theta = \frac{5}{2}$$
$$\theta = 68.2°$$

$\overrightarrow{AB}$ has magnitude 5.39 units and its direction makes an angle $-68.2°$ with the x-axis.

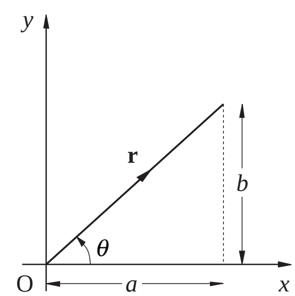


The convention used to define direction in the example above is that angles are measured **positive anticlockwise** from $0x$ up to and including $180°$ and **negative clockwise** from $0x$ up to, but not including, $-180°$.

How many other ways can you find of uniquely defining the direction of a vector?

In general for a vector $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$ its magnitude is given by

$$|\mathbf{r}| = \sqrt{a^2 + b^2}$$



and its direction is given by θ where

$$\cos\theta = \frac{a}{|\mathbf{r}|} \quad \sin\theta = \frac{b}{|\mathbf{r}|} \quad \text{and} \quad \tan\theta = \frac{b}{a}.$$

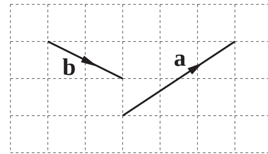
The angle θ can be found from one of these together with a sketch. The vector $\mathbf{0} = 0\mathbf{i} + 0\mathbf{j}$ has magnitude zero and is called the zero vector.

Adding vectors

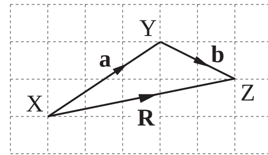
Vectors have magnitude and direction. To complete the definition of a vector, it is necessary to know how to add two vectors.

When vectors are added, it is equivalent to one displacement followed by another.

When $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$ is added to $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$, it is the same as displacement $\overrightarrow{XY}$ followed by displacement $\overrightarrow{YZ}$. From the diagrams opposite,



$$\begin{aligned} \mathbf{R} &= \mathbf{a} + \mathbf{b} \\ &= (3\mathbf{i} + 2\mathbf{j}) + (2\mathbf{i} - \mathbf{j}) \\ &= (3+2)\mathbf{i} + (2-1)\mathbf{j} \\ &= 5\mathbf{i} + \mathbf{j} \end{aligned}$$



The single vector **R** is equivalent to $\mathbf{a} + \mathbf{b}$, with **R**, **a** and **b** forming a triangle XYZ.

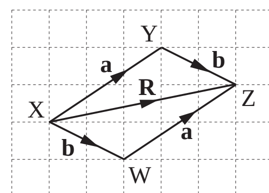
The components are added independently of each other. **R** is called the **resultant** of **a** and **b** and this property of vectors is called the **triangle law of addition**.

In general, in component form, if $\mathbf{p} = d\mathbf{i} + e\mathbf{j}$ and $\mathbf{q} = f\mathbf{i} + g\mathbf{j}$ then $\mathbf{p} + \mathbf{q} = (d+f)\mathbf{i} + (e+g)\mathbf{j}$

Adding vectors can be considered in terms of a parallelogram law as well as a triangle law. In fact, the parallelogram law includes the triangle law

$$\overrightarrow{XZ} = \overrightarrow{XY} + \overrightarrow{YZ}$$

or $\qquad \mathbf{R} = \mathbf{a} + \mathbf{b}$.



From the lower triangle XWZ the result is obviously just as valid. This shows that the resultant vector **R** of **a** and **b** is either **a** followed by **b** or **b** followed by **a**.

> A vector is any quantity possessing the properties of magnitude and direction, which obeys the triangle law of addition

Activity 3 Resultant vectors

Find the resultant vector $\mathbf{a} + \mathbf{b}$

(a) by drawing;

(b) by adding components;

for the following values of **a** and **b**:

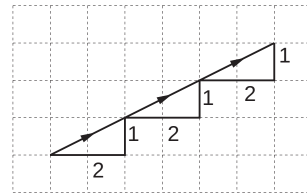
(i) $\mathbf{a} = \mathbf{i} + 3\mathbf{j} \quad \mathbf{b} = 4\mathbf{i} + 2\mathbf{j}$

(ii) $\mathbf{a} = -2\mathbf{i} + \mathbf{j} \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j}$

(iii) $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} \quad \mathbf{b} = \mathbf{i} + \mathbf{j}$

(iv) $\mathbf{a} = -3\mathbf{i} + \mathbf{j} \quad \mathbf{b} = -2\mathbf{i} - 4\mathbf{j}$

Multiplication by a scalar



Suppose a displacement $2\mathbf{i}+\mathbf{j}$ is repeated three times. This is equivalent to adding three equal vectors:

$$(2\mathbf{i}+\mathbf{j})+(2\mathbf{i}+\mathbf{j})+(2\mathbf{i}+\mathbf{j}).$$

The result is $6\mathbf{i}+3\mathbf{j}$ or $3(2\mathbf{i}+\mathbf{j})$, and the scale factor of 3 scales each component separately.

The vector $2\mathbf{i}+\mathbf{j}$ is said to have been multiplied by the scalar 3. In general, for 2 non-zero vectors **a** and **b**, if $\mathbf{a}=s\mathbf{b}$ where s is a scalar, then **a** is **parallel** to **b**.

If $s>0$, **a** and **b** are in the same direction, but if $s<0$ then **a** and **b** are in opposite directions.

In general, in component form, if vector **a** is given as

$$\mathbf{a}=x\mathbf{i}+y\mathbf{j}$$

and s is a scalar, then the vector $s\mathbf{a}$ is

$$\begin{aligned} s\mathbf{a} &= s(x\mathbf{i}+y\mathbf{j}) \\ &= sx\mathbf{i}+sy\mathbf{j} \end{aligned}$$

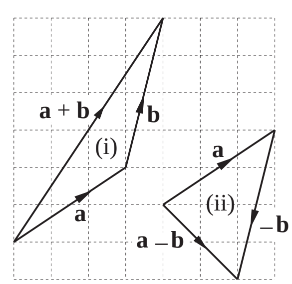
Subtracting vectors

You have seen so far that vectors can be added and multiplied by scalars. What then of subtracting vectors?

Take $\mathbf{a}=3\mathbf{i}+2\mathbf{j}$ and $\mathbf{b}=\mathbf{i}+4\mathbf{j}$

Operating as with addition,

$$\begin{aligned} \mathbf{a}-\mathbf{b} &= \mathbf{a}+(-\mathbf{b}) \\ &= 3\mathbf{i}+2\mathbf{j}+(-\mathbf{i}-4\mathbf{j}) \\ &= (3-1)\mathbf{i}+(2-4)\mathbf{j} \\ &= 2\mathbf{i}-2\mathbf{j} \end{aligned}$$



Geometrically, $-\mathbf{b}$ is equal in magnitude but opposite in direction to **b**, so while $\mathbf{a}+\mathbf{b}$ is shown in (i) on the right, $\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$ is shown in (ii).

You can see that both addition and subtraction of vectors use the triangle law of addition.

Exercise 3B

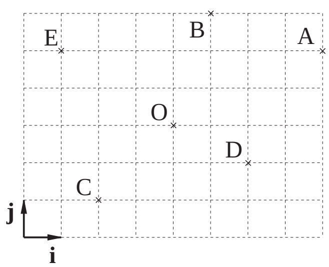
1. Write in the form $a\mathbf{i}+b\mathbf{j}$ the vectors:

   (a) $\overrightarrow{OA}$ (b) $\overrightarrow{OB}$ (c) $\overrightarrow{AB}$

   (d) $\overrightarrow{BA}$ (e) $\overrightarrow{BC}$ (f) $\overrightarrow{CD}$

   (g) $\overrightarrow{BD}$ (h) $\overrightarrow{CE}$ (i) $\overrightarrow{DA}$

   (j) $\overrightarrow{EA}$ (k) $\frac{1}{2}\overrightarrow{EC}$ (l) $5\overrightarrow{CA}$.



2. In the diagram for question 1:

   (a) the point Q has position vector $3\mathbf{i}+\mathbf{j}$ Find the vectors $\overrightarrow{QO}$, $\overrightarrow{QC}$, $\overrightarrow{DQ}$;

   (b) the point R has position vector $p\mathbf{i}+q\mathbf{j}$ Find in terms of p and q, $\overrightarrow{RO}$, $\overrightarrow{RC}$ and $\overrightarrow{AR}$.

3. The triangle law of addition for $\overrightarrow{AB}$ can be verified using triangle OAB from the diagram of question 1:-

$$\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}$$

L.H.S. $=\overrightarrow{AB}$
$=-3\mathbf{i}+\mathbf{j}$

R.H.S. $=\overrightarrow{AO}+\overrightarrow{OB}$
$=(-4\mathbf{i}-2\mathbf{j})+(\mathbf{i}+3\mathbf{j})$
$=(-4\mathbf{i}+\mathbf{i})+(-2\mathbf{j}+3\mathbf{j})$
$=-3\mathbf{i}+\mathbf{j}$

Hence L.H.S. = R.H.S. showing $\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}$.

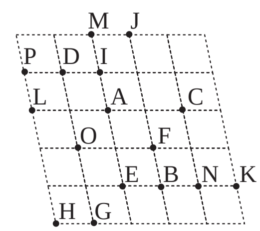
Verify the triangle law of addition for $\overrightarrow{CD}$ using triangle OCD.

4. Using the grid shown, write down as many vectors as you can that are equal to:

   (a) $\overrightarrow{OA}$ (b) $\overrightarrow{EB}$

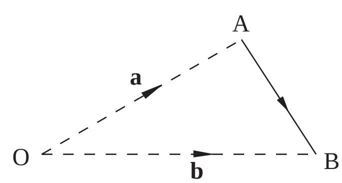
   (c) $\overrightarrow{BO}$ (d) $\overrightarrow{EI}$

   (e) $\overrightarrow{FL}$ (f) $\overrightarrow{GP}$.



3.2 Relative position vectors

The position vector of B relative to A is simply $\overrightarrow{AB}$. Then, using the triangle law of addition,



$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO}+\overrightarrow{OB}\\ &= -\mathbf{a}+\mathbf{b}\\ &= \mathbf{b}-\mathbf{a}.\end{aligned}$$

where **a** and **b** are the position vectors of A and B, respectively, relative to the origin O.

Example

The point P has position vector $-\mathbf{i}+\mathbf{j}$ the point Q, $\mathbf{i}+6\mathbf{j}$ and the point R, $-\mathbf{j}$ Find the magnitudes and directions of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.

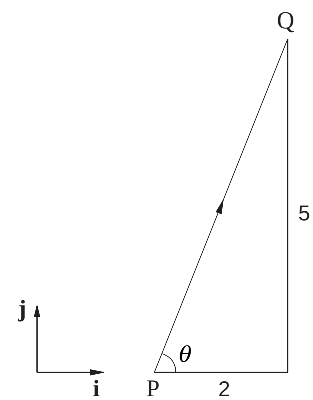
Solution

$$\begin{aligned}\vec{PQ} &= \vec{PO} + \vec{OQ}\\ &= -\vec{OP} + \vec{OQ}\\ &= \vec{OQ} - \vec{OP}\\ &= (\mathbf{i} + 6\mathbf{j}) - (-\mathbf{i} + \mathbf{j})\\ &= 2\mathbf{i} + 5\mathbf{j}\end{aligned}$$

The magnitude of $\vec{PQ}$ is given by

$$|\vec{PQ}| = \sqrt{2^2 + 5^2}$$

$$= \sqrt{29} = 5.39.$$



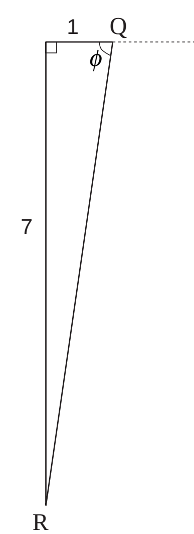
The direction is given by

$$\tan\theta = \frac{5}{2}$$

so that $\theta = 68.2°.$

Similarly

$$\begin{aligned}\vec{QR} &= \vec{OR} - \vec{OQ}\\ &= -\mathbf{j} - (\mathbf{i} + 6\mathbf{j})\\ &= -\mathbf{i} - 7\mathbf{j}.\end{aligned}$$



The magnitude of $\vec{QR}$ is given by

$$|\vec{QR}| = \sqrt{(-1)^2 + (-7)^2}$$

$$= \sqrt{50} = 7.07.$$

The direction of $\vec{QR}$ is $-180° + \phi$, where

$$\tan\phi = \frac{7}{1}$$

giving $\phi = 81.9°.$

So the direction of $\vec{QR}$ is $-180° + 81.9° = -98.1°.$

Exercise 3C

1. Find the magnitudes and directions of:

   $\mathbf{a} = 3\mathbf{i} - 9\mathbf{j}$ $\quad \mathbf{b} = -2\mathbf{i} - \mathbf{j}$

   $\mathbf{c} = \sqrt{3}\mathbf{i} + 3\mathbf{j}$ $\quad \mathbf{d} = -5\mathbf{i} + 4\mathbf{j}$

2. Points P, Q, R and S have position vectors

   $\mathbf{p} = 2\mathbf{i} - \mathbf{j}$ $\mathbf{q} = -\mathbf{i} + \mathbf{j}$ $\mathbf{r} = 2\mathbf{j}$ and $\mathbf{s} = a\mathbf{i} + 6\mathbf{j}$

   Find the magnitudes of $\overrightarrow{PQ}$, $\overrightarrow{PR}$, $\overrightarrow{QR}$ and $\overrightarrow{PS}$.

   Given that $\left|\overrightarrow{PS}\right| = \sqrt{50}$, find the possible values of a.

3. Points A to F have position vectors **a** to **f** respectively, defined in terms of **p**, **q** and **r** as follows:

   $\mathbf{a} = \mathbf{p} + \mathbf{q} + \mathbf{r}$, $\quad \mathbf{b} = \mathbf{p} + \mathbf{q} - \mathbf{r}$, $\quad \mathbf{c} = 2\mathbf{p}$,

   $\mathbf{d} = 3\mathbf{q} - \mathbf{r}$, $\quad \mathbf{e} = -\mathbf{p} + 4\mathbf{q}$, $\quad \mathbf{f} = \frac{1}{2}(\mathbf{p} + \mathbf{q})$.

   Find in terms of **p**, **q** and **r**, the vectors

   (a) $\overrightarrow{AB}$ (b) $\overrightarrow{BC}$ (c) $\overrightarrow{AC}$
   (d) $\overrightarrow{EC}$ (e) $\overrightarrow{BD}$ (f) $\overrightarrow{FA}$
   (g) $\overrightarrow{DF}$ (h) $\overrightarrow{CE}$ (i) $\overrightarrow{ED}$
   (j) $\overrightarrow{BF}$.

3.3 Unit vectors

You should be getting used to using **i** and **j** which are unit vectors in the directions O*x* and O*y*. A **unit vector** is simply a vector having **magnitude one**, and can be in **any direction**. To find a unit vector in the direction of $\mathbf{c} = 3\mathbf{i} - 4\mathbf{j}$ you multiply by a scalar, so that its direction is unchanged but its magnitude is altered to one.

The magnitude of **c** is

$$|\mathbf{c}| = \sqrt{3^2 + (-4)^2}$$
$$= 5$$

which is five times as big as the magnitude of a unit vector. **c** must be multiplied by $\frac{1}{5}$ or divided by 5 to make a unit vector in its direction.

A unit vector in the direction of **c** is

$$\frac{1}{5}(3\mathbf{i} - 4\mathbf{j})$$
$$= \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

In general, if a vector **a** has magnitude $|\mathbf{a}|$, then a unit vector in the direction of **a** is denoted $\hat{\mathbf{a}}$ and

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

Equal vectors

If the vectors $c\mathbf{i}+d\mathbf{j}$ and $e\mathbf{i}+f\mathbf{j}$ are equal, then

$$c\mathbf{i}+d\mathbf{j}=e\mathbf{i}+f\mathbf{j}$$

and it follows that

$$c=e \text{ and } d=f.$$

These are the only possible conclusions if the vectors are equal. Note that $c=e$ comes from equating the **i** components of the equal vectors and $d=f$ comes from equating the **j** components.

You will see in later chapters that the technique of 'equating components' is very useful in the solution of problems.

Example

Vectors **p** and **q** are defined in terms of x and y as

$$\mathbf{p}=3\mathbf{i}+(y-2)\mathbf{j} \text{ and } \mathbf{q}=2x\mathbf{i}-7\mathbf{j}$$

If $\mathbf{p}=2\mathbf{q}$, find the values of x and y.

Solution

Since $\mathbf{p}=2\mathbf{q}$,

$$3\mathbf{i}+(y-2)\mathbf{j}=2(2x\mathbf{i}-7\mathbf{j})$$

giving $3\mathbf{i}+(y-2)\mathbf{j}=4x\mathbf{i}-14\mathbf{j}$

Equating **i** components gives

$$3=4x$$

and so

$$x=\frac{3}{4}.$$

Equating **j** components gives

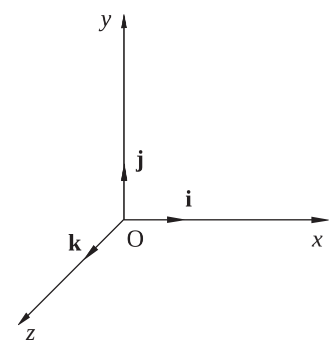
$$y-2=-14$$

and so

$$y=-12.$$

3.4 Vectors in three dimensions

The results obtained so far have all been applied to vectors in one or two dimensions. However, the power of vectors is that they can be applied in one, two or three dimensions. Although the applications of mechanics in this book will be restricted to one or two dimensions, by taking a vector approach, the extensions to three dimensional applications will be easier.



In order to work in three dimensions, it is necessary to define a third axis Oz, so that Ox, Oy and Oz form a right-handed set as in the diagram opposite. An ordered trio of numbers such as (2, 3, 4) is necessary to define the coordinates of a point and a vector must have 3 components. For example, the position vector of the point (2, 3, 4) is $2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ where **k** is a unit vector in the direction Oz.

The properties of vectors considered so far are all defined in three dimensions as the following examples show.

Example

If $\mathbf{p}=3\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $\mathbf{q}=\mathbf{i}+3\mathbf{j}-2\mathbf{k}$, find $\mathbf{p}+\mathbf{q}$ and $4\mathbf{q}$.

Solution

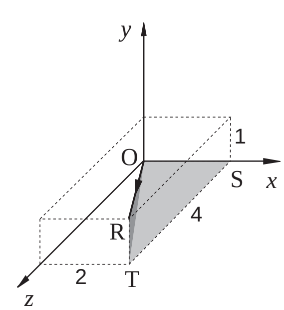
$$\begin{aligned}
\mathbf{p}+\mathbf{q} &= (3\mathbf{i}-2\mathbf{j}+\mathbf{k})+(\mathbf{i}+3\mathbf{j}-2\mathbf{k}) \\
&= (3+1)\mathbf{i}+(-2+3)\mathbf{j}+(1-2)\mathbf{k} \\
&= 4\mathbf{i}+\mathbf{j}-\mathbf{k}. \\
4\mathbf{q} &= 4(\mathbf{i}+3\mathbf{j}-2\mathbf{k}) \\
&= 4\mathbf{i}+12\mathbf{j}-8\mathbf{k}.
\end{aligned}$$

Example

The point R has position vector $\mathbf{r}=2\mathbf{i}+\mathbf{j}+4\mathbf{k}$. Find OR.

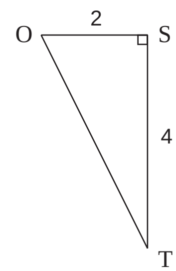
Solution

From the diagram, it can be seen that OR is the diagonal of a cuboid with dimensions 2, 1, 4 units.



Using Pythagoras' theorem, OR can be found from the right angled triangles OST, OTR.

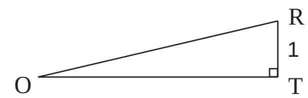
From triangle OST,



$$OT^2 = OS^2 + ST^2$$
$$= 2^2 + 4^2$$
$$= 4 + 16$$

giving $OT = \sqrt{20}.$

From triangle OTR,



$$OR^2 = OT^2 + RT^2$$
$$= 20 + 1^2$$

giving $OR = \sqrt{21} = 4.58.$

Note that $OR = |\mathbf{r}|$ and that extending the two dimensional result for modulus of a vector gives

$$|r| = \sqrt{2^2 + 1^2 + 4^2}$$
$$= \sqrt{4 + 1 + 16}$$
$$= \sqrt{21} = 4.58.$$

In general, if $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

then $|\mathbf{r}| = \sqrt{a^2 + b^2 + c^2}.$

Example

The points A, B and C have position vectors $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + \mathbf{k}$, $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$. Find $\overrightarrow{AB}$, $|\overrightarrow{BC}|$ and the unit vector in the direction of $\overrightarrow{BC}$.

Solution

The vector $\overrightarrow{AB}$ is given by

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$
$$= (-2\mathbf{i} + \mathbf{k}) - (\mathbf{i} - \mathbf{j} + 3\mathbf{k})$$
$$= -3\mathbf{i} + \mathbf{j} - 2\mathbf{k}.$$

$$\vec{BC} = \mathbf{c} - \mathbf{b}$$
$$= (3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) - (-2\mathbf{i} + \mathbf{k})$$
$$= 5\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}.$$
$$|\vec{BC}| = \sqrt{5^2 + 2^2 + (-5)^2}$$
$$= \sqrt{54} = 7.35$$

The unit vector in the direction of BC is $\frac{1}{\sqrt{54}}(5\mathbf{i} + 2\mathbf{j} - 5\mathbf{k})$

$$= \frac{5}{\sqrt{54}}\mathbf{i} + \frac{2}{\sqrt{54}}\mathbf{j} - \frac{5}{\sqrt{54}}\mathbf{k}.$$

Exercise 3D

1. If $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} - 4\mathbf{k}$, find:
   (a) $\mathbf{a} + \mathbf{b} + \mathbf{c}$;
   (b) $5\mathbf{a}$;
   (c) $2\mathbf{b} + 3\mathbf{c}$.
2. If $\mathbf{p} = \mathbf{i} + \mathbf{j} + 5\mathbf{k}$, $\mathbf{q} = 6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{r} = -3\mathbf{j} + 4\mathbf{k}$ are the position vectors of the points P, Q and R, find:
   (a) the position vector $\vec{PQ}$;
   (b) $\vec{QR}$;
   (c) $|\vec{PR}|$;
   (d) the unit vector in the direction $\vec{QP}$.
3. If the vectors $4\mathbf{i} - m\mathbf{j}$ and $(3n-2)\mathbf{i} + (3n+2)\mathbf{j}$ are equal in magnitude but opposite in direction, find the values of m and n.
4. Vectors **a** and **b** are defined in terms of x, y and z as
   $$\mathbf{a} = \mathbf{i} + (x+y)\mathbf{j} - (2x+y)\mathbf{k}$$
   $$\mathbf{b} = z\mathbf{i} + (3y+2)\mathbf{j} - (5x-3)\mathbf{k}.$$
   If $\mathbf{a} = \mathbf{b}$ then find the values of x, y and z and show that
   $$\left|\frac{5}{4}x\mathbf{i} + 5y\mathbf{j} + 6z\mathbf{k}\right| = \sqrt{46}.$$

3.5 Scalar products

So far, vectors have been added, subtracted and multiplied by a scalar. Just as the addition of two vectors is a different operation from the addition of two real numbers, the product of two vectors has its own definition.

The **scalar product** of two vectors, **a** and **b**, is defined as $ab\cos\theta$, where θ is the angle between the two vectors, and **a** and **b** are the moduli (or magnitude) of **a** and **b**. The scalar product is usually written as **a.b**, read as “a dot b”, so

$$\mathbf{a}.\mathbf{b} = ab\cos\theta.$$

If the vectors **a** and **b** are perpendicular the scalar product **a.b** is zero since $\cos 90° = 0$.

So

$$\mathbf{a}.\mathbf{b} = 0 \text{ when } \mathbf{a} \text{ is perpendicular to } \mathbf{b}.$$

Also, if the vectors **a** and **b** are parallel, the scalar product **a.b** is given by *ab*, since $\cos 0 = 1$.

So

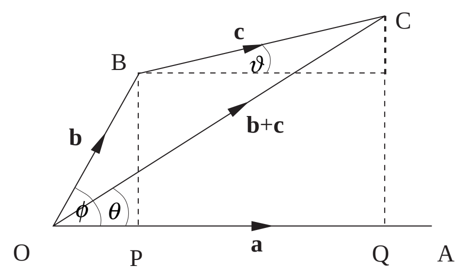
$$\mathbf{a}.\mathbf{b} = ab \text{ when } \mathbf{a} \text{ is parallel to } \mathbf{b}.$$

The scalar product follows both the commutative and distributive laws.

$$\begin{aligned} \mathbf{a}.\mathbf{b} &= ab\cos\theta \\ &= ba\cos\theta \\ &= \mathbf{b}.\mathbf{a} \end{aligned}$$

This shows that the scalar product is commutative ie $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$.

The scalar product **a** .(**b**+**c**) can be found by considering the diagram opposite.



Since $OQ = OP + PQ$

then $|\mathbf{b}+\mathbf{c}|\cos\theta = b\cos\phi + c\cos\vartheta$.

Multiplying by a $(=|\mathbf{a}|)$ gives

$$|\mathbf{a}||\mathbf{b}+\mathbf{c}|\cos\theta = ab\cos\phi + ac\cos\vartheta$$

So $\qquad \mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$

This result shows that the scalar product is distributive over addition.

Calculating the scalar product

To calculate the scalar product of two vectors you must find the magnitude of each vector and the angle between the vectors. The magnitude of a vector $\mathbf{p} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is given by

$$|p| = \sqrt{a^2 + b^2 + c^2}\,.$$

Since **i**, **j** and **k** are perpendicular unit vectors, the scalar product gives some valuable results.

Perpendicular vectors: $\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = \mathbf{i}.\mathbf{k} = \mathbf{k}.\mathbf{j} = \mathbf{j}.\mathbf{i} = 0$

Parallel vectors: $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$

These results are used to find **p.q**

where $\mathbf{p} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

and $\mathbf{q} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$

$$\begin{aligned}\mathbf{p}.\mathbf{q} &= (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}).(d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) \\ &= ad\,\mathbf{i}.\mathbf{i} + ae\,\mathbf{i}.\mathbf{j} + af\,\mathbf{i}.\mathbf{k} \\ &\quad + bd\,\mathbf{j}.\mathbf{i} + be\,\mathbf{j}.\mathbf{j} + bf\,\mathbf{j}.\mathbf{k} \\ &\quad + cd\,\mathbf{k}.\mathbf{i} + ce\,\mathbf{k}.\mathbf{j} + cf\,\mathbf{k}.\mathbf{k} \\ &= ad + 0 + 0 + 0 + be + 0 + 0 + 0 + cf\end{aligned}$$

$$\mathbf{p}.\mathbf{q} = ad + be + cf \qquad (1)$$

But $\mathbf{p}.\mathbf{q} = pq\cos\theta$

So $\cos\theta = \dfrac{\mathbf{p}.\mathbf{q}}{pq} = \dfrac{ad + be + cf}{pq}$ (2)

Example

If $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $b = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ find

a.b and the angle between **a** and **b.**

Solution

$$a = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$b = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$

Using equation (1)

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= 1\times 2 + 1\times 3 + 1\times 4 \\ &= 9\end{aligned}$$

and equation (2)

$$\cos\theta = \frac{9}{\sqrt{3}\sqrt{29}}$$

$$\theta = 15.2°$$

Exercise 3E

1. Show that the vectors $2\mathbf{i}+5\mathbf{j}$ and $15\mathbf{i}-6\mathbf{j}$ are perpendicular.
2. Show that the vectors $4\mathbf{i}-3\mathbf{j}+\mathbf{k}$ and $2\mathbf{i}+3\mathbf{j}+\mathbf{k}$ are perpendicular.
3. If $\mathbf{a}=2\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-\mathbf{j}+\mathbf{k}$, find **a.b** and the angle between **a** and **b**.
4. If two forces are described by the vectors $\mathbf{i}-2\mathbf{j}+2\mathbf{k}$ and $2\mathbf{i}+2\mathbf{j}+\mathbf{k}$ find the cosine of the angle between the forces.

3.6 Velocity as a vector

Velocity is the rate of change of displacement, a measure of how the position of an object is changing with time.

Contrast this with speed, which is the rate of change of distance travelled with respect to time.

To compare these quantities, consider a child on a roundabout in a playground, being pushed by her father. He pushes slowly but steadily so that it takes 4 seconds for one revolution. The circumference of the roundabout is 12 metres.

For one revolution, the child's average speed is

$$\frac{12 \text{ metres}}{4 \text{ seconds}} = 3 \text{ ms}^{-1}.$$

But after one revolution, the child's displacement from her initial position is **0**. (She is back to where she started.)for one revolution, her average velocity is

$$\frac{\mathbf{0}}{4} = \mathbf{0}.$$

The magnitude of her average velocity is 0 ms^{-1}.

The magnitude of the average velocity and the average speed are equal only when the motion is in a straight line, with no changes of direction.

If an object changes from position **r** to position **s** in a time t, then the average velocity is given by

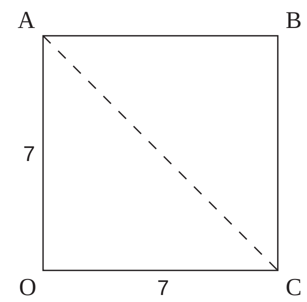
$$\text{average velocity} = \frac{\text{change in displacement}}{\text{time}}$$

or

$$\mathbf{v} = \frac{\mathbf{s}-\mathbf{r}}{t} \text{ or } \frac{1}{t}\mathbf{s} - \frac{1}{t}\mathbf{r}.$$

Because this is the difference of two scaled displacement Because this is the difference of two scaled displacement vectors, each multiplied by a scalar, it is a vector quantity.

For example, Sally and Floella use $\mathsf{v} = \dfrac{\mathsf{s} - \mathsf{r}}{t}$ to calculate a known average velocity to verify that the result works. Their classroom is a square 7 metres by 7 metres and the intention is that one of the girls walks the diagonal at steady speed. They will then calculate her average velocity both directly and from $\mathsf{v} = \dfrac{\mathsf{s} - \mathsf{r}}{t}$ so that the results can be compared.



They use Pythagoras' theorem to calculate the length of the diagonal and find it is approximately 10 metres.

Sally walks the diagonal AC at steady speed. Floella times her at 10 seconds. So they know her average speed is $\dfrac{10}{10} = 1\ \text{ms}^{-1}$. Taking the origin as O, OC as x-axis and OA as y-axis, her direction is $-45°$ to the x-axis. So they take her average velocity to be $1\ \text{ms}^{-1}$ at $-45°$ to Ox.

Sally and Floella take the magnitude of the average velocity to be the average speed. Why?

This is their verification:-

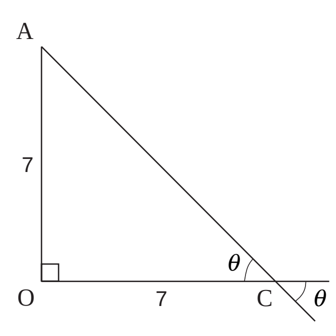
since the position vector of A is 7**j**

and the position vector of C is 7**i**,

the average velocity $\mathsf{v} = \dfrac{7\mathsf{i} - 7\mathsf{j}}{10} \left(\text{using } \dfrac{\mathsf{s} - \mathsf{r}}{t}\right)$.

The magnitude of **v** is

$$|\mathsf{v}| = \frac{\sqrt{7^2 + 7^2}}{10} = \frac{\sqrt{98}}{10} \approx 1\ \text{ms}^{-1}$$



and its direction is given by $-\theta$ to Ox where

$$\tan\theta = \frac{7}{7}$$

giving $\theta = 45°$.

So using $\mathsf{v} = \dfrac{\mathsf{s} - \mathsf{r}}{t}$ gives them the same result as the known average velocity, i.e. $1\ \text{ms}^{-1}$ at $-45°$ to Ox.

3.7 Acceleration as a vector

Acceleration is the rate of change of velocity with time. If an object changes from a velocity **v** to a velocity **w** in a time *t*, then average acceleration is

$$\text{average acceleration} = \frac{\text{change in velocity}}{\text{time}}$$

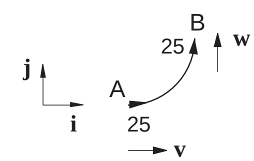
or

$$\mathbf{a} = \frac{\mathbf{w} - \mathbf{v}}{t}.$$

This is also a vector quantity.

Example

A car travels round a bend which forms a quadrant of a circle at a constant speed of 25 ms^{-1}. If the bend takes 5 seconds to negotiate, find the average acceleration of the car during this period.

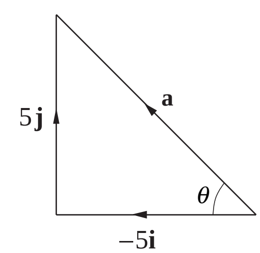


Solution

Define unit vectors **i**, **j** as shown.

The velocity of the car at A is $\mathbf{v} = 25\mathbf{i}$ and at B is $\mathbf{w} = 25\mathbf{j}$

so the average acceleration from A to B is



$$\mathbf{a} = \frac{\mathbf{w} - \mathbf{v}}{t} = \frac{25\mathbf{j} - 25\mathbf{i}}{5}$$

$$= -5\mathbf{i} + 5\mathbf{j}$$

The magnitude of **a** is

$$|\mathbf{a}| = \sqrt{5^2 + 5^2} = \sqrt{50} = 7.07,$$

and its direction is $180 - \theta$ where

$$\tan\theta = 1, \qquad \theta = 45°.$$

Note that the speed of the car is not changing as it negotiates the bend, but the velocity is changing.

Exercise 3F

1. Verify the result:

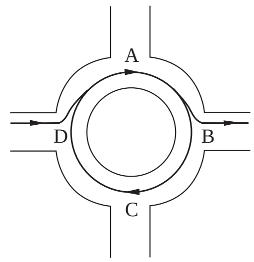
   average velocity $= \dfrac{\mathbf{s}-\mathbf{r}}{t}$

   based on walking the diagonal of a rectangular room 6 metres by 8 metres in 8 seconds.

2. The driver of a car is unsure of his route. He approaches a roundabout intending to turn right, but changes his mind, eventually doing $1\frac{1}{2}$ circuits of the roundabout and going straight on.

   If he travels at a constant 12 ms^{-1} and takes 16 seconds for one complete circuit ABCDA, then calculate his average acceleration for the $\frac{1}{4}$ circle AB. Hence write down the average acceleration for $\frac{1}{4}$ circles BC, CD, DA.

   Calculate the average acceleration for the $\frac{1}{2}$ circle CDA.



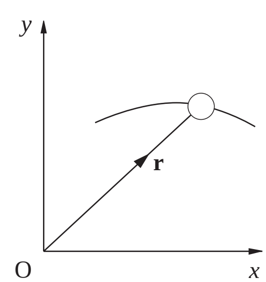
Discuss what you would understand to be the average acceleration for a complete circle of the roundabout in Question 2.

3.8 Instantaneous velocity and acceleration

You have seen that for a one dimensional motion, if the displacement x from a fixed point O is known as a function of time $x = f(t)$ then the velocity and acceleration at any time t can be found by successive differentiation:

$$\text{velocity} = \frac{dx}{dt} \qquad \text{acceleration} = \frac{d^2x}{dt^2}.$$

One dimensional motion is not very common however. The motion of a tennis ball during a game, or a child on a swing or a jumping frog takes place in two or three dimensions.



If the position vector **r** of a netball passed between 2 players is known, you can calculate the average velocity and average acceleration. But what about velocity and acceleration at an instant of time t?

If **r** is known as a function of t, can this be successively differentiated? If so, can the same meanings be attached to these derivatives as in the one dimensional case?

Two kinematics activities revisited

In your study of one dimensional kinematics, the motion of a rolling ball was investigated in two activities, 'Galileo's experiment' and the 'chute experiment'. The motion of a ball rolling down an incline was investigated in the first of these activities, and in the second, the motion of a ball along a level horizontal surface.

In 'Galileo's experiment', you will have obtained a relationship of the form

$$y = kt^2 \qquad (3)$$

where y is the distance in metres rolled by the ball in time t seconds, and k is a constant. This is a simple quadratic function model for a ball rolling down an incline and your k was probably about 0.1.

Choosing $k = 0.1$, equation (3) becomes

$$y = 0.1t^2.$$

Differentiating this gives the velocity, $\frac{dy}{dt}$, at time t,

$$\frac{dy}{dt} = 2(0.1)t$$
$$\frac{dy}{dt} = 0.2t.$$

Differentiating again gives the acceleration, $\frac{d^2y}{dt^2}$, at time t,

$$\frac{d^2y}{dt^2} = 0.2$$

This is constant, independent of time.

In the 'chute experiment', you will have obtained a relationship of the form

$$x = ct \qquad (4)$$

where x is the distance in metres rolled by the ball in time t seconds, and c is a constant. This is a simple linear function model for a ball rolling along a level horizontal surface and your value of c was probably about 0.5.

Choosing $c = 0.5$, equation (4) becomes

$$x = 0.5t.$$

Differentiating this gives the velocity $\frac{dx}{dt}$, at time t

$$\frac{dx}{dt} = 0.5 \qquad \text{(constant velocity).}$$

Differentiating again gives the acceleration, $\frac{d^2x}{dt^2}$, at time t

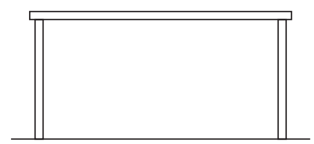
$$\frac{d^2x}{dt^2} = 0 \qquad \text{(zero acceleration).}$$

Activity 4 Combining 'Galileo's experiment' with the 'chute experiment'.

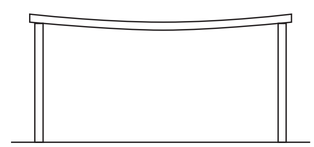
You will need the following for this activity:

level table, a chute, billiard ball, stopwatch, two blocks to incline the table.

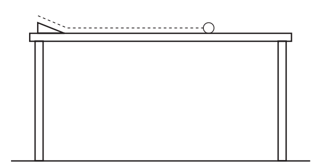
Stage 1



Choose a level table



Not bowed in the middle



Set up the chute experiment. Mark a point on the chute from which the ball takes about 2 seconds to roll the length of the table. Make sure you **always** release the ball from this mark in stages 1 and 3.

Take readings of time for **one** distance only, say 1 metre, for several releases of the ball. Average the results. Use this one pair of readings for x and t to substitute into the model $x = ct$ and find c, rounding your value to one decimal place. For example, if your values are

$$x = 1 \text{ metres}, \quad t = 1.45 \text{ seconds}$$

then substituting into $x = ct$ gives

$$1 = 1.45c$$

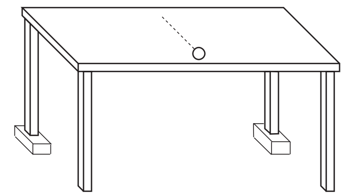
$$\Rightarrow \quad c = \frac{1}{1.45}$$

$$\Rightarrow \quad c = 0.7 \qquad \text{(1 decimal place).}$$

Stage 2

Incline the table so that the slope runs down its width and the ball takes about 2 seconds to roll down the width. Now do 'Galileo's experiment' as follows.

Take readings of time for **one** distance only for the ball to roll from rest down the width of the table. Average the results. Use this one pair of readings for y and t to substitute into the model $y = kt^2$ and find k, rounding your value to one decimal place. For example, if your values are



$$y = 0.6 \text{ m} \qquad t = 2.26 \text{ s}$$

then substituting into $y = kt^2$ gives

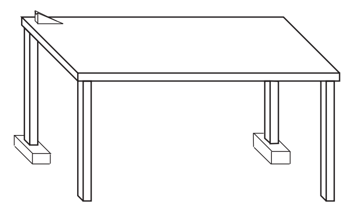
$$0.6 = k(2.26)^2$$

$$\Rightarrow \qquad k = \frac{0.6}{(2.26)^2}$$

$$\Rightarrow \qquad k = 0.1 \qquad \text{(1 decimal place).}$$

Stage 3

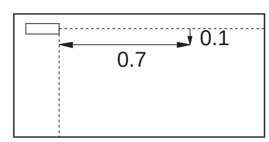
Leaving the table inclined exactly as in stage 2, set up the chute at the top left hand side of the table, but pointing **along** its length.



Before releasing the ball down the chute, predict its path using the values of c and k from stages 1 and 2.

If you have

$$x = 0.7t$$



and

$$y = 0.1t^2,$$

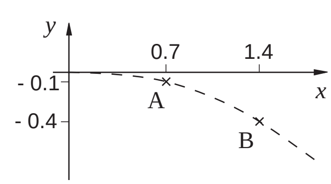
then, when $t = 1$ second

$$x = 0.7(1) = 0.7$$

and $$y = 0.1\left(1^2\right) = 0.1 \text{ m.}$$

This predicts the coordinates of the position of the ball after one second.

Make 4 or 5 predictions using your own values of c and k. Mark them lightly on the table in chalk and join them with a smooth curve.

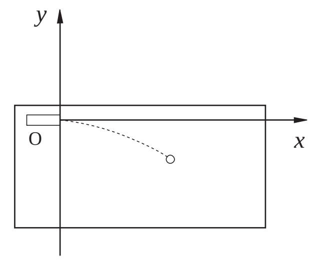


Having marked the predictions, release the ball from your mark on the chute and see how accurate your predictions are.

Release the ball a number of times; you should consistently obtain a curved path close to that predicted.

Position vector of the ball

Choose axes as shown in the diagram opposite. They are in the plane of the table. Ox is along the length of the table in line with the chute. Oy is perpendicular to Ox but pointing up the incline so that the origin O is at the foot of the chute.



For the data given as an example in Activity 4, the path of the ball is predicted by the set of coordinates $(x, y) = (0.7t, -0.1t^2)$, or by the position vector

$$\mathbf{r} = 0.7t\,\mathbf{i} - 0.1t^2\,\mathbf{j}. \tag{5}$$

This shows that the two one dimensional motions of Stages 1 and 2 have been combined to produce a single motion in two dimensions in Stage 3.

What is the path predicted by your data?

3.9 Investigating the velocity and speed of the ball

The average velocity of the ball in Activity 4 can be found between any two points on its path using the result of Section 3.6,

$$\text{average velocity } \mathbf{v} = \frac{\mathbf{s} - \mathbf{r}}{t}.$$

For the average velocity between $t = 1$ and $t = 2$ seconds,

when $t = 1$

$$\mathbf{r} = 0.7(1)\mathbf{i} - 0.1(1^2)\mathbf{j} \text{ from equation (5)}$$
$$= 0.7\mathbf{i} - 0.1\mathbf{j}$$

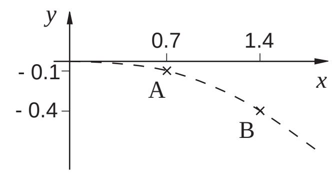
when $t = 2$

$$\mathbf{s} = 0.7(2)\mathbf{i} - 0.1(2^2)\mathbf{j}$$
$$= 1.4\mathbf{i} - 0.4\mathbf{j}$$

So the average velocity

$$= \frac{(1.4\mathbf{i} - 0.4\mathbf{j}) - (0.7\mathbf{i} - 0.1\mathbf{j})}{2 - 1}$$
$$= 0.7\mathbf{i} - 0.3\mathbf{j}$$

The diagram opposite shows the positions A and B of the ball when $t = 1$ second and when $t = 2$ seconds.



As well as the velocity of the ball between these two times, the approximate average speed can be calculated using

$$\text{approximate average speed} = \frac{\text{straight line distance AB}}{\text{time taken}}$$

$$= \frac{\sqrt{0.7^2 + 0.3^2}}{2-1}$$

$$= 0.76 \text{ ms}^{-1}.$$

What are the average velocity and approximate average speed between $t = 1$ and $t = 2$ seconds for your data from Activity 4?

Activity 5 Calculating average velocity

For the ball whose path is given by equation (5), find the average velocity and approximate average speed between the following time intervals.

time intervals	average velocity	approximate average speed
$t = 1$ to $t = 2$	$0.7\mathbf{i} - 0.3\mathbf{j}$	0.76
$t = 1$ to $t = 1.5$		
$t = 1$ to $t = 1.2$		
$t = 1$ to $t = 1.1$		
$t = 1$ to $t = 1.05$		

What do you notice about your results? If you were asked to predict the velocity and speed of the ball at the point A, when $t = 1$ second, what would you predict? Write down both of your predictions.

Why is the average speed used in Activity 5 approximate?

What could you have measured to make it exact?

How accurate do you consider the approximation to be?

Velocity and speed at a point on the path or at an instant of time *t*

Can a meaning now be given to differentiating the position vector

$$\mathbf{r} = 0.7t\mathbf{i} - 0.1t^2\mathbf{j}?$$

Since **i** and **j** are fixed unit vectors, they are constants and are treated as such when differentiating.

So $$\frac{d\mathbf{r}}{dt} = 0.7\mathbf{i} - 2(0.1)t\mathbf{j}$$

giving $$\frac{d\mathbf{r}}{dt} = 0.7\mathbf{i} - 0.2t\mathbf{j} \qquad (6)$$

Since the position vector is a function of *t*, differentiating it is simply differentiating its components as functions of *t*.

Substituting $t = 1$ in equation (6) gives

$$\frac{d\mathbf{r}}{dt} = 0.7\mathbf{i} - 0.2\mathbf{j}$$

whose magnitude is

$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{0.7^2 + 0.2^2} = 0.73.$$

Compare these values with your predictions of velocity and speed at $t = 1$ from Activity 5. You should find

$\frac{d\mathbf{r}}{dt}$ is the velocity of the ball at $t = 1$,

$\left|\frac{d\mathbf{r}}{dt}\right|$ is the speed of the ball at t = 1.

Example

A ball is rolled on a table and axes chosen so that its position vector at time *t* seconds is given in metres by

$$\mathbf{r} = \left(1 - 0.2t^2\right)\mathbf{i} + 0.6t\mathbf{j}$$

Find its velocity and speed after 2 s.

Solution

$$\mathbf{r} = \left(1 - 0.2t^2\right)\mathbf{i} + 0.6t\mathbf{j}.$$

Differentiating with respect to *t* gives

$$\frac{d\mathbf{r}}{dt} = -0.4t\,\mathbf{i} + 0.6\,\mathbf{j}$$

When $t = 2$

$$\frac{d\mathbf{r}}{dt} = -0.8\mathbf{i} + 0.6\,\mathbf{j}$$

and $$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{(0.8)^2 + 0.6^2} = 1.$$

So velocity is $-0.8\mathbf{i} + 0.6\,\mathbf{j}\,\text{ms}^{-1}$, and the speed is 1 ms^{-1}.

Example

The position vector of a clay pigeon t seconds after release is given in metres by

$$\mathbf{r} = 3t\,\mathbf{i} + \left(30t - 5t^2\right)\mathbf{j} + 40t\,\mathbf{k}.$$

Calculate its velocity and speed after one second.

Solution

Differentiating **r** with respect to t gives

$$\frac{d\mathbf{r}}{dt} = 3\mathbf{i} + (30 - 10t)\,\mathbf{j} + 40\,\mathbf{k}\,.$$

When $t = 1$

$$\frac{d\mathbf{r}}{dt} = 3\mathbf{i} + 20\,\mathbf{j} + 40\,\mathbf{k}$$

and

$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{3^3 + 20^2 + 40^2}$$

$$= 44.8\ \text{ms}^{-1}.$$

After one second, its velocity is $3\mathbf{i} + 20\,\mathbf{j} + 40\,\mathbf{k}\ \text{ms}^{-1}$ and its speed is $44.8\ \text{ms}^{-1}$.

Exercise 3G

1. The position vector at time t seconds of a football struck from a free kick is given in metres by:

   $\mathbf{r} = 24t\,\mathbf{i} + \left(7t - 5t^2\right)\mathbf{j}$

   Find its velocity at time t and determine the speed with which the ball was struck.

2. The position vector of an aircraft flying horizontally at time t seconds is given in metres by:

   $\mathbf{r} = 120t\,\mathbf{i} + 160t\,\mathbf{j}$

   where **i** is directed east and **j** north. What is the speed of the aircraft? On what bearing is it heading?

3. The position vector of a golf ball t seconds after it has been struck is given in metres by:

$$\mathbf{r} = 60t\,\mathbf{i} + \left(12t - 5t^2\right)\mathbf{j} - t\,\mathbf{k}.$$

Find its speed and its velocity after 2.4 seconds.

4. During the first three seconds of her fall, the position vector of a sky-diver is given in metres by:

$$\mathbf{r} = 100t\,\mathbf{i} + \left(1000 - 5t^2\right)\mathbf{j} - 30t\,\mathbf{k}.$$

Find her velocity after 2 seconds and speed after 2.5 seconds.

3.10 Average acceleration of the ball

For the combined Galileo/chute experiment the average acceleration of the ball over a second is the change in velocity in that second.

Activity 6 Calculating average acceleration

For the ball whose velocity is given in equation (6) by

$$\frac{d\mathbf{r}}{dt} = 0.7\mathbf{i} - 0.2t\,\mathbf{j}$$

assume that the table is large enough for the motion to take place over 5 s.

Complete these values of velocity:

time	**velocity** $\frac{d\mathbf{r}}{dt}$
$t = 0$	$0.7\mathbf{i} - 0\,\mathbf{j}$
$t = 1$	$0.7\mathbf{i} - 0.2\,\mathbf{j}$
$t = 2$	$0.7\mathbf{i} - 0.4\,\mathbf{j}$
$t = 3$	
$t = 4$	
$t = 5$	

Now use the result of Section 3.7,

$$\text{average acceleration } \mathbf{a} = \frac{\mathbf{w} - \mathbf{v}}{t}$$

to complete the following values:

time interval	average acceleration
$t = 0$ to $t = 1$	$\dfrac{(0.7\mathbf{i} - 0.2\mathbf{j}) - (0.7\mathbf{i} - 0\mathbf{j})}{1 - 0} = 0\mathbf{i} - 0.2\mathbf{j}$
$t = 1$ to $t = 2$	$\dfrac{(0.7\mathbf{i} - 0.4\mathbf{j}) - (0.7\mathbf{i} - 0.2\mathbf{j})}{2 - 1} =$
$t = 2$ to $t = 3$	
$t = 3$ to $t = 4$	
$t = 4$ to $t = 5$	

What do these results suggest about the acceleration of the ball?

____

Acceleration at a point on the path or at an instant of time *t*

Differentiating the velocity vector $\dfrac{d\mathbf{r}}{dt} = 0.7\mathbf{i} - 0.2t\,\mathbf{j}$ gives

$$\frac{d^2\mathbf{r}}{dt^2} = 0\mathbf{i} - 0.2\mathbf{j}$$

which should agree with the values you obtained in Activity 6.

This suggests that the second derivative $\dfrac{d^2\mathbf{r}}{dt^2}$ of position vector **r** represents acceleration. The acceleration is constant in this case because $0\mathbf{i} - 0.2\mathbf{j}$ is independent of t.

The comparison with the one-dimensional case is now complete.

In two dimensions, starting with position vector

$$\mathbf{r} = 0.7t\mathbf{i} - 0.1t^2\mathbf{j}$$

differentiating once gives the velocity vector

$$\frac{d\mathbf{r}}{dt} = 0.7\mathbf{i} - 0.2t\,\mathbf{j}$$

and differentiating a second time gives the acceleration vector

$$\frac{d^2\mathbf{r}}{dt^2} = 0\mathbf{i} - 0.2\mathbf{j}$$

Summary

If the position vector, **r**, of an object is known as a function of time, t, then the instantaneous velocity $\frac{d\mathbf{r}}{dt}$ and the instantaneous acceleration, $\frac{d^2\mathbf{r}}{dt^2}$ as functions of t can be calculated from **r** by differentiation:

if $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$,

then $$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$$

and $$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$$

Example

The position vector of a golf ball t seconds after it has been struck is given in metres by:

$$\mathbf{r} = 50t\mathbf{i} + \left(14t - 5t^2\right)\mathbf{j} + 2t\mathbf{k}.$$

Find its speed when $t = 2.5$ and show that the acceleration is constant and in the negative y direction. What is its magnitude?

Solution

$$\mathbf{r} = 50t\mathbf{i} + \left(14t - 5t^2\right)\mathbf{j} + 2t\mathbf{k}$$

Differentiating **r** with respect to t gives

$$\frac{d\mathbf{r}}{dt} = 50\mathbf{i} + (14 - 10t)\mathbf{j} + 2\mathbf{k}.$$

When $t = 2.5$, the velocity is

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= 50\mathbf{i} + (14 - 25)\mathbf{j} + 2\mathbf{k} \\ &= 50\mathbf{i} - 11\mathbf{j} + 2\mathbf{k}\end{aligned}$$

and the speed is

$$\begin{aligned}\left|\frac{d\mathbf{r}}{dt}\right| &= \sqrt{50^2 + 11^2 + 2^2} \\ &= 51.2 \text{ ms}^{-1}.\end{aligned}$$

Differentiating again gives the acceleration

$$\frac{d^2\mathbf{r}}{dt^2} = 0\mathbf{i} - 10\mathbf{j} + 0\mathbf{k}.$$

This is a constant acceleration of magnitude 10 ms^{-2} in the negative y direction.

Exercise 3H

1. The position vector of a netball at time t seconds is given in metres by

$$\mathbf{r} = 12t\,\mathbf{i} + \left(16t - 5t^2\right)\mathbf{j}$$

where **i** is horizontal and **j** is vertically upwards. Determine the velocity with which it was thrown and show that its acceleration is constant. What is the direction of this acceleration?

2. Determine the accelerations in each of Questions 3 and 4 from Exercise 3G. What conclusion can you draw about each of these motions?

3. The velocity of a particle is

$$\frac{d\mathbf{r}}{dt} = (3t+1)\mathbf{i} - 4t\,\mathbf{j}$$

What is its acceleration at time t?

3.11 Miscellaneous Exercises

1. (a) Find the magnitude and direction of the following vectors:

   (i) $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j}$ (ii) $\mathbf{b} = \mathbf{i} + \sqrt{3}\mathbf{j}$

   (iii) $\mathbf{c} = -\mathbf{i} + 3\mathbf{j}$

   (b) Find the magnitude of the following vectors:

   (i) $\mathbf{d} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ (ii) $\mathbf{e} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

   (iii) $\mathbf{f} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$.

2. For the vectors in Question 1, find a unit vector in the direction of each vector.

3. The points A and B have position vectors

   $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{j}$

   Find:

   (a) the vector $\overrightarrow{AB}$;

   (b) the magnitude of vector $\overrightarrow{OA}$;

   (c) the unit vector in the direction of $\overrightarrow{OB}$.

4. The vectors **p** and **q** are defined in terms of a, b and c as:

   $$\mathbf{p} = a\,\mathbf{i} + (a-b)\mathbf{j} + c\mathbf{k}$$

   and $\mathbf{q} = (5-b)\mathbf{i} + (2b+7)\mathbf{j} + (a+5b)\mathbf{k}$.

   If $\mathbf{p} = \mathbf{q}$, find the values of a, b and c. Determine the unit vector in the direction of $\mathbf{p} - \mathbf{s}$ where $\mathbf{s} = 5\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$.

5. Given the vectors in Question 1, find :

   (a) $\mathbf{a} + 2\mathbf{b}$ (b) $2\mathbf{c} - 3\mathbf{d}$

   (c) $\mathbf{a} + \mathbf{d} - 2\mathbf{f}$ (d) the vector **x** if $2\mathbf{a} - \mathbf{x} = \mathbf{e}$.

6. Given the vectors

   $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{j}$

   (a) describe the direction of $\mathbf{a} + \mathbf{b}$;

   (b) find the vectors $2\mathbf{a} + 3\mathbf{b} - \mathbf{c}$ and $-\mathbf{a} + 2\mathbf{b} + \mathbf{c}$;

   (c) P is the end-point of the displacement vector $2\mathbf{a} + 3\mathbf{b} - \mathbf{c}$ and $(1, -2, 3)$ is the starting point.

   What is the position vector of P?

7. Given the vectors

   $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{c} = \mathbf{i} - \mathbf{j} - \mathbf{k}$,

   (a) find the magnitude and describe the direction of $\mathbf{a} + \frac{1}{2}\mathbf{b}$;

   (b) find the vectors $3\mathbf{a} - \mathbf{b} + \mathbf{c}$ and $-2\mathbf{a} + \mathbf{b} - 3\mathbf{c}$.

   (c) P is the end-point of the displacement vector $-2\mathbf{a} + \mathbf{b} - 3\mathbf{c}$ and $(-1, 0, -2)$ is its starting point. What is the position vector of P?

8. Referred to the x and y axes, the coordinates of the points P and Q are (3, 1) and (4, –3) respectively.

(a) Calculate the magnitude of the vector $\overrightarrow{OP} + 5\,\overrightarrow{OQ}$;

(b) Calculate the magnitude and direction of the vector $\overrightarrow{PQ}$;

(c) Calculate the coordinates of the point R if $\overrightarrow{OP} + \overrightarrow{OQ} = 2\,\overrightarrow{OR}$.

9. The velocity of a particle is given at time t seconds, as

$$\mathbf{v} = 6t\,\mathbf{i} - \mathbf{j} + t^2\mathbf{k}.$$

Find its acceleration at time t and its speed after 3s.

10. The position vector of a particle at time t seconds is

$$\mathbf{r} = t\,\mathbf{i} + \left(\frac{1}{t}\right)\mathbf{j}$$

What is its velocity at time t and speed after 2 s?

11. The coordinates of a moving point P at time t seconds are $\left(4t^2,\ 8t\right)$ metres.

(a) Write down the position vector of P.

(b) Find the velocity of P.

(c) Show that the acceleration of P is always parallel to the x axis.

12. At time t, the position vectors of two points, P and Q, are given by:

$$\mathbf{p} = 2t\,\mathbf{i} + \left(3t^2 - 4t\right)\mathbf{j} + t^3\mathbf{k}$$

$$\mathbf{q} = t^3\mathbf{i} - 2t\,\mathbf{j} + \left(2t^2 - 1\right)\mathbf{k}.$$

Find the velocity and acceleration of Q relative to P when $t = 3$.

13. The position vector of smoke particles as they leave a chimney for the first 4 s of their motion is given by

$$\mathbf{r} = 4t\,\mathbf{i} + \left(\frac{3t^2}{2}\right)\mathbf{j} + 6t\,\mathbf{k},$$

where **i** and **k** are horizontal, directed north and east respectively, and **j** is vertically upward.

(a) What is the magnitude and direction of the acceleration of the smoke?

(b) What is (i) the velocity
and (ii) the speed
of the smoke particles after 2 s?

(c) In what direction does the smoke go relative to the **i**, **k** plane (i.e. relative to the ground)?

14. A glider spirals upwards in a thermal (hot air current) so that its position vector with respect to a point on the ground is

$$\mathbf{r} = \left(100\cos\frac{t}{5}\right)\mathbf{i} + \left(200 + \frac{t}{3}\right)\mathbf{j} + \left(100\sin\frac{t}{5}\right)\mathbf{k}.$$

The directions of **i**, **j** and **k** are as defined in Question 13.

(a) Determine the glider's speed at $t = 0,\ 5\pi$ and 10π seconds.
What do you notice?

(b) Find **r** when $t = 0$ and 10π seconds and find the height risen in one complete turn of the spiral.

15. A particle P moves in such a way that its position vector **r** at time t is given by

$$\mathbf{r} = t^2\mathbf{i} + (t+1)\,\mathbf{j} + t^3\mathbf{k}.$$

(a) Find the velocity and acceleration vectors.

(b) Find a unit vector along the direction of the tangent to the path of the motion.

16. A particle of mass 3 kg moves in a horizontal plane and its position vector at time t s relative to a fixed origin O is given by

$$\mathbf{r} = \left(2\sin t\,\mathbf{i} + \cos t\,\mathbf{j}\right)\text{ m}.$$

Find the values of t in the range $0 \le t \le \pi$ when the speed of the particle is a maximum.

(AEB)

17. Two particles P and Q have velocities $\left(3\mathbf{i} - 5\mathbf{j}\right)\text{ ms}^{-1}$ and $\left(2\mathbf{i} - 3\mathbf{j}\right)\text{ ms}^{-1}$ respectively. The line of motion of P passes through the point A with position vector $\left(5\mathbf{i} + 13\mathbf{j}\right)$ m, relative to a fixed origin O, and the line of motion of Q passes through the point B with position vector $\left(7\mathbf{i} + 9\mathbf{j}\right)$ m relative to O.

In the case when P and Q pass through the points A and B respectively at the same time, find the velocity of P relative to Q and deduce that the particles will collide two seconds after passing through these points. Find also the position vector, relative to O, of the point of collision. Given that the particles have equal mass and stick together upon collision, find the velocity of the combined mass after collision.

(AEB)

*18. Two particles P and Q are moving on a horizontal plane and at time t seconds

$$\mathbf{OP} = -a\left(\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j}\right), \qquad \mathbf{OQ} = (vt - a)\,\mathbf{i},$$

where a, v, ω are constants and O is a fixed point in the plane. Show that both P and Q are moving with constant speeds.

Denoting the speed of P by u show that the square of the speed of P relative to Q is,

$$v^2 + u^2 - 2vu\sin\omega t.$$

When $t = 0$ the speed of P relative to Q is $\sqrt{80}$ ms^{-1} and when $t = \dfrac{\pi}{6\omega}$, the speed of P relative to Q is $\sqrt{48}$ ms^{-1}. Given that P is moving faster than Q, find the speeds of P and Q.

(AEB)

19. A particle moves in a plane and at time t its position vector, **r**, is given by

$$\mathbf{r} = (2\cos t)\mathbf{i} + (\sin t)\mathbf{j}$$

Find the values of t in the range $0 \le t \le \pi$ when

(a) the speed of the particle is a maximum,

(b) the force acting on the particle is perpendicular to the velocity.

(AEB)

20. A particle P of mass 0.25 kg moves on a smooth horizontal table with constant velocity $(17\mathbf{i} + 6\mathbf{j})$ ms^{-1}, where **i** and **j** are perpendicular constant unit vectors in the plane of the table. An impulse is then applied to the particle so that its velocity becomes $(29\mathbf{i} + 22\mathbf{j})$ ms^{-1}. Find this impulse in the form of $a\mathbf{i} + b\mathbf{j}$

Determine a unit vector **n** such that the component of the velocity of P along **n** is unchanged by the impulse. Obtain the magnitude of this component.

(AEB)

4 VECTORS 2

Objectives

After studying this chapter you should

- be able to integrate acceleration vectors to obtain velocity and position vectors;
- be able to understand the consequences of modelling force as a vector in equilibrium and non-equilibrium situations;
- be able to resolve forces into perpendicular components;
- be able to apply the law of friction in its simplest form;
- be able to appreciate and use Newton's Second Law.

4.1 From acceleration to velocity to position vector

In the last chapter you used differentiation to derive the velocity vector **v** and acceleration vector **a** from a given position (i.e. displacement) vector **r**. Suppose, instead, that you are given the acceleration **a** of an object moving in a plane, for example

$$\mathbf{a} = 2\mathbf{i} + 3t\mathbf{j}.$$

This means

$$\frac{d^2\mathbf{r}}{dt^2} = 2\mathbf{i} + 3t\mathbf{j}.$$

Integrating with respect to t gives

$$\frac{d\mathbf{r}}{dt} = (2t + A)\mathbf{i} + \left(\frac{3t^2}{2} + B\right)\mathbf{j}$$

where A and B are some constants. In order to determine the velocity vector precisely, more information is needed: for instance, the velocity at a particular time, say

$$\mathbf{v} = 3\mathbf{i} + \frac{1}{2}\mathbf{j} \text{ when } t = 1.$$

Then $$3\mathbf{i} + \frac{1}{2}\mathbf{j} = (2 \times 1 + A)\mathbf{i} + \left(\frac{3}{2} \times 1^2 + B\right)\mathbf{j} \quad .$$

Equating components gives

$$3 = 2 + A,\ \frac{1}{2} = \frac{3}{2} + B$$

so that $A = 1$ and $B = -1$. Then

$$\mathbf{v} = (2t+1)\mathbf{i} + \left(\frac{3}{2}t^2 - 1\right)\mathbf{j}.$$

Integrating again

$$\mathbf{r} = \left(t^2 + t + C\right)\mathbf{i} + \left(\frac{1}{2}t^3 - t + D\right)\mathbf{j}$$

for some constants C and D.

Again, specific information is needed in order to identify C and D. Suppose that, when $t = 1$, $\mathbf{r} = \mathbf{0}$ (i.e. the object is at the origin).

Then $$0\mathbf{i} + 0\mathbf{j} = (1 + 1 + C)\mathbf{i} + \left(\frac{1}{2} - 1 + D\right)\mathbf{j}.$$

Equating components gives $0 = 2 + C,\ 0 = -\frac{1}{2} + D$

so that $$C = -2 \text{ and } D = \frac{1}{2}.$$

Then $$\mathbf{r} = \left(t^2 + t - 2\right)\mathbf{i} + \left(\frac{1}{2}t^3 - t + \frac{1}{2}\right)\mathbf{j}.$$

In general,

$$\boxed{\mathbf{r} = x\mathbf{i} + y\mathbf{j}} \ \underset{\leftarrow \text{ integrate}}{\text{differentiate} \rightarrow}\ \boxed{\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}} \ \underset{\leftarrow \text{ integrate}}{\text{differentiate} \rightarrow}\ \boxed{\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}}$$

The values of the velocity and displacement at the start of the motion, i.e. at time $t = 0$, are needed to determine the constants which arise in the integration and are called the **initial conditions**.

Example

A particle starts from the origin with velocity $2\mathbf{i} + \mathbf{j}$, and is subject to an acceleration of $\mathbf{i} - 3\mathbf{j}$. Find the velocity and position vectors of the particle after t seconds, expressed as functions of t.

Solution

Integrating the acceleration

$$\mathbf{a} = \mathbf{i} - 3\mathbf{j}$$

gives the velocity

$$\mathbf{v} = (t + A)\mathbf{i} + (-3t + B)\mathbf{j},$$

where A and B are some constants.

When $t = 0$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$

so $$2\mathbf{i} + \mathbf{j} = (0 + A)\mathbf{i} + (0 + B)\mathbf{j}$$

and equating components gives $A = 2$, $B = 1$

so $\mathbf{v} = (t + 2)\mathbf{i} + (1 - 3t)\mathbf{j}$ is the velocity vector.

Integrating the velocity gives the position vector

$$\mathbf{r} = \left(\frac{t^2}{2} + 2t + C\right)\mathbf{i} + \left(t - \frac{3}{2}t^2 + D\right)\mathbf{j}$$

where C and D are constants.

When $t = 0$, $\mathbf{r} = \mathbf{0}$

so $$0\mathbf{i} + 0\mathbf{j} = (0 + 0 + C)\mathbf{i} + (0 + 0 + D)\mathbf{j}$$

and equating components gives

$$C = D = 0.$$

Thus $$\mathbf{r} = \left(\frac{t^2}{2} + 2t\right)\mathbf{i} + \left(t - \frac{3t^2}{2}\right)\mathbf{j}$$

is the position vector.

Example

A particle starts moving from the point $2\mathbf{j}$ with initial velocity $3\mathbf{i} - \mathbf{k}$ under an acceleration of $6t\,\mathbf{j}$ at time t seconds after the motion begins. Find the velocity and position vectors of the particle as functions of time.

Solution

Notice here that you are now working in three dimensions; however, the same rules apply.

$$\mathbf{a} = 6t\,\mathbf{j}$$

Integrating gives

$$\mathbf{v} = A\mathbf{i} + \left(3t^2 + B\right)\mathbf{j} + C\,\mathbf{k}\ .$$

When $t = 0$, $\mathbf{v} = 3\mathbf{i} - k$

so $\quad 3\mathbf{i} - \mathbf{k} = A\mathbf{i} + (0 + B)\mathbf{j} + C\mathbf{k}$.

Equating components gives

$$A = 3,\ B = 0,\ C = -1.$$

Then $\quad \mathbf{v} = 3\mathbf{i} + 3t^2\mathbf{j} - \mathbf{k}$ is the velocity vector.

Integrating again

$$\mathbf{r} = (3t + D)\mathbf{i} + \left(t^3 + E\right)\mathbf{j} + (-t + F)\mathbf{k}.$$

When $t = 0$, $\mathbf{r} = 2\mathbf{i}$,

so $\quad 2\mathbf{i} = (0 + D)\mathbf{i} + (0 + E)\mathbf{j} + (0 + F)\mathbf{k}$.

Equating components gives

$$D = 2,\ E = F = 0.$$

Then $\quad \mathbf{r} = (3t + 2)\mathbf{i} + t^3\mathbf{j} - t\mathbf{k}$ is the position vector.

Exercise 4A

1. Given the position vector of a particle, find its velocity and acceleration vectors as functions of time t:

   (a) $\mathbf{r} = 2t\,\mathbf{i} + \left(3 - 5t^3\right)\mathbf{j}$

   (b) $\mathbf{r} = \mathbf{i} - 3t\,\mathbf{k}$

   (c) $\mathbf{r} = \dfrac{1}{t^2}\mathbf{i} + 4t\,\mathbf{j} + \dfrac{1}{2}t^2\mathbf{k}$

2. Given the acceleration vector $\mathbf{a} = -0.2\,\mathbf{j}$ of an object, with initial velocity $0.5\mathbf{i}$, and that the object starts at the origin, find its velocity and position vectors at any time t.

3. A particle P is moving such that at time t seconds it has position vector

   $$\mathbf{r} = -t\,\mathbf{i} + 8t^2\,\mathbf{j} + \left(1 + t - t^2\right)\mathbf{k} \text{ metres.}$$

   Show that P has a constant acceleration, stating its magnitude. Find also the speed of P when $t = 5\,\text{s}$.

4. In each case (a) to (c), find the velocity and position vectors, given the acceleration vector of a particle and the initial conditions.

   (a) $\mathbf{a} = 5\mathbf{i} - t\,\mathbf{k}$; when $t = 0$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 7\mathbf{k}$

   (b) $\mathbf{a} = 4\sqrt{t}\,\mathbf{j} - \dfrac{1}{2}t^2\,\mathbf{k}$; when $t = 0$, $\mathbf{v} = \dfrac{1}{3}\mathbf{j}$, $\mathbf{r} = -\mathbf{i} + \dfrac{5}{3}\mathbf{j} - \mathbf{k}$

   (c) $\mathbf{a} = -10\,\mathbf{j}$; when $t = 0$, $\mathbf{v} = 3\mathbf{i} + 2\,\mathbf{j}$, $\mathbf{r} = 0$

5. For Question 4, parts (a) and (b), find the speed of the particle when $t = 2$ and the distance of the particle from the origin at this time.

4.2 Force as a vector

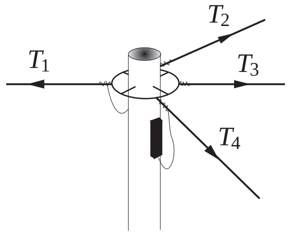
Up until now you have seen the vectorial nature of displacement, velocity and acceleration. However, the principal application of vector properties will be in the modelling of forces.

Activity 1 Identifying forces

Identify as many situations as you can, that you could expect to find in and around the home, where several forces are acting on an object.

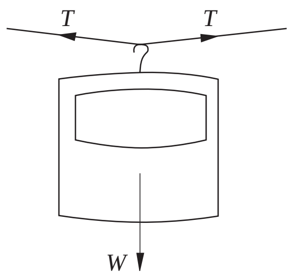
In each case, try to specify the natures of the various forces involved; where possible give a brief qualitative analysis of these forces.

Here are some further examples for you to consider.



1. The top of a telegraph pole

The 'pulling' forces in each wire (tensions) are most probably of different strengths, hence labelled differently here.



2. The hanger of a peg-bag on a clothes line

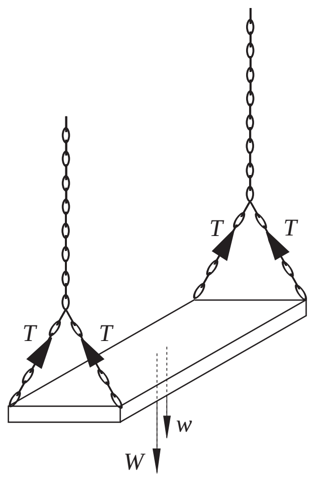
Here, each portion of the line exerts a tension on the hanger. Note that they are labelled the same since you might reasonably expect the tension to be the same throughout the entire length of the line. There is also the weight of the pegs, bag, and the hanger itself, pulling the hanger downwards. (These have all been combined into the single weight W).

3. The seat of a garden-swing with a child on it

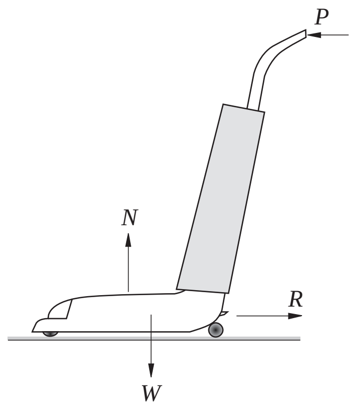
The tensions in the four portions of chain are labelled the same since, if the child sits centrally, you can expect these to be equal.

W = weight of child

w = weight of seat



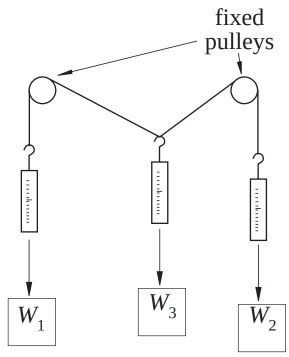
4. A vacuum cleaner 'hoovering' the carpet



P is a pushing force applied to the cleaner via the handle, of variable magnitude and changing direction from time to time, N is the normal contact force of the floor on the vacuum cleaner, W is its weight, R resistances to motion, including friction.

If forces are vectors, then they need to satisfy the parallelogram law.

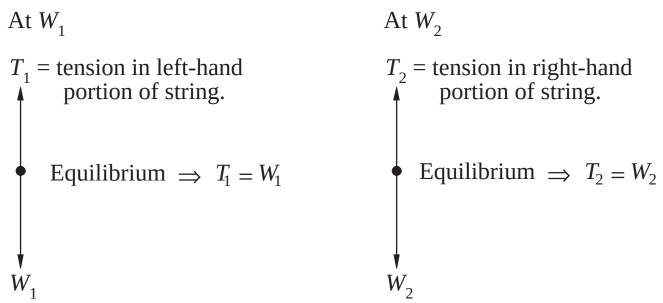
Activity 2 Three-force experiment



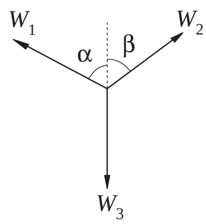
You will need a simple kit including two pulleys, some string, weights and a sheet of paper for this activity.

Set the pulley system up as shown in the diagram:

W_1, W_2 and W_3 are (different) small weights attached to the string so that the system is at rest.

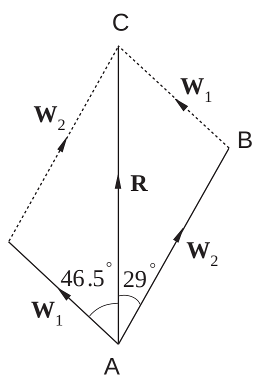


At the point of application of the three forces involved, the forces are shown in the diagram opposite.



Choose different values for the weights W_1, W_2 and W_3 to hang on the system at the three points, and measure the angles α and β as accurately as you can. For each set of values, verify the following result by means of a scale-drawing:

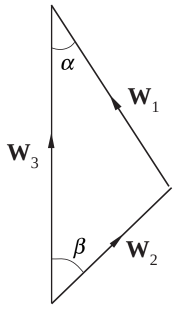
e.g.	W_1	α	W_2	β	W_3
	20	46.5°	30	29°	40



The resultant of the vectors $\mathbf{W}_1$ and $\mathbf{W}_2$, denoted by **R**, can be found by the **parallelogram rule**, and measurement of the scale-drawing shows that **R** has the same magnitude as the vector $\mathbf{W}_3$, but opposite direction; which is to be expected if the point of application is in equilibrium.

Did you find any sets of weights which would not remain in equilibrium: Can you explain why not?

When two forces act at a point which does not move, the two forces must be equal and opposite. For example, think of the forces acting on a book lying on the table. Activity 2 shows that when three forces act at a point which does not move, the resultant of two of them formed by the parallelogram law is equal and opposite to the third, indicating that forces must add as vectors. If on your scale drawing of Activity 2 you replace the resultant of $\mathbf{W}_1$ and $\mathbf{W}_2$ by the equal and opposite force $\mathbf{W}_3$, then $\mathbf{W}_1$, $\mathbf{W}_2$ and $\mathbf{W}_3$ are the sides of a triangle of forces.

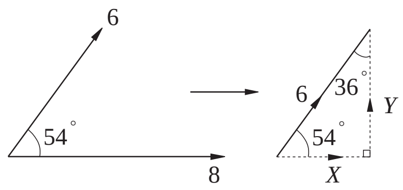


4.3 Resolving forces

Any vector quantity, such as force, can be written in terms of its perpendicular components. This process of writing a force as the sum of its components is called **resolving**, or resolution. Breaking a force down into its resolved parts is important in simplifying two-or three-dimensional situations.

Example

Two forces acting at $54°$ to each other have magnitudes 6 N and 8 N. Find the components of the smaller force parallel and perpendicular to the larger one.

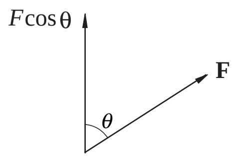


Solution

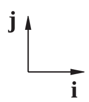
The component of the 6 N force parallel to the 8 N force, X, is $6\cos 54°$; while the component of the 6 N force perpendicular to the 8 N force, Y, is $6\cos 36°$.

In general the component of a force **F** in a given direction is

(the magnitude of **F**) $\times$ (the cosine of the angle turned through)



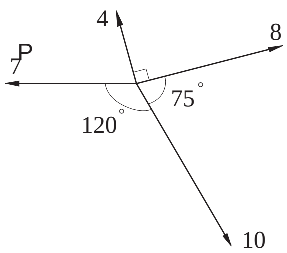
In terms of vectors, it is easier to specify unit vectors **i** and **j** (the reference system) in the chosen directions and then write each force in **i**, **j** form. In this example, calling the 8 N force **F** and the 6N force **G** and defining **i**, **j** as shown,



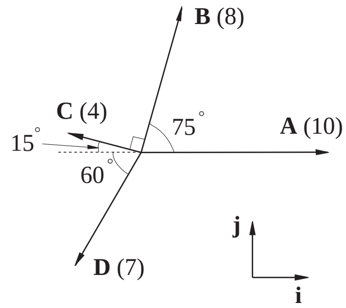
$$\mathbf{F} = 8\mathbf{i} \text{ and } \mathbf{G} = 6\cos 54°\,\mathbf{i} + 6\cos 36°\,\mathbf{j}$$

Example

Four forces act on a particle P. Write each force in **i**, **j** form, where **i** is in the direction of the largest force, and **j** is perpendicular to it in an anti-clockwise sense. (Note that when units are not specified on a force diagram, they are newtons).



Firstly, it may help to label the forces and re-draw them as shown opposite, marking in suitable angles.



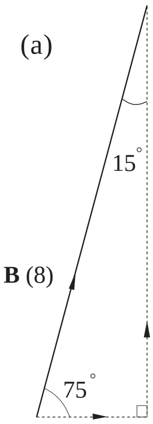
Then $\mathbf{A} = 10\mathbf{i}$

(a) $\mathbf{B} = 8\cos 75°\,\mathbf{i} + 8\cos 15°\,\mathbf{j}$

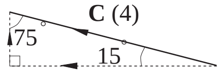
(b) $\mathbf{C} = -4\cos 15°\,\mathbf{i} + 4\cos 75°\,\mathbf{j}$

(c) $\mathbf{D} = -7\cos 60\,\mathbf{i} - 7\cos 30°\,\mathbf{j}$

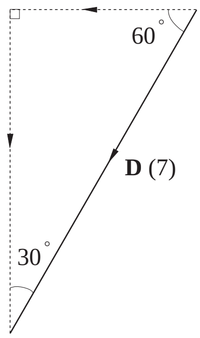
(a)



(b)



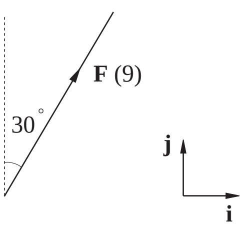
(c)



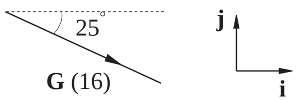
Exercise 4B

1. Write each force in **i**, **j** form:

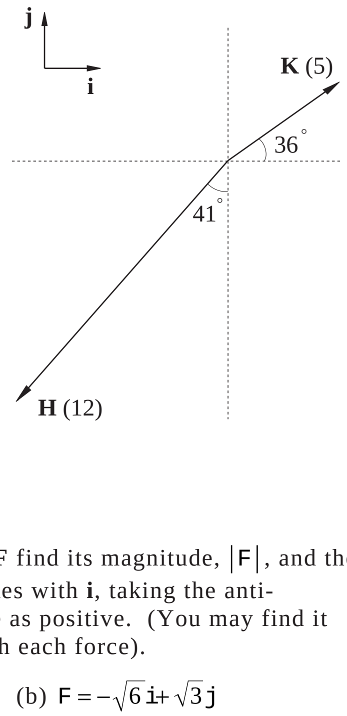
(a)



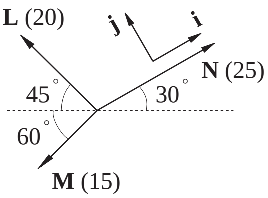
(b)



(c)



(d)



2. A force $\mathbf{F} = a\,\mathbf{i} + 12\,\mathbf{j}$ acts at $20°$ to **i**. Find $|\mathbf{F}|$ (the magnitude of **F**) and hence the value of a.

3. For each force **F** find its magnitude, $|\mathbf{F}|$, and the direction it makes with **i**, taking the anti-clockwise sense as positive. (You may find it helpful to sketch each force).

(a) $\mathbf{F} = 8\mathbf{i} + 3\mathbf{j}$ (b) $\mathbf{F} = -\sqrt{6}\,\mathbf{i} + \sqrt{3}\,\mathbf{j}$

(c) $\mathbf{F} = 20\mathbf{i} - 21\mathbf{j}$ (d) $\mathbf{F} = 7\sin 27°\,\mathbf{i} + 7\cos 27°\,\mathbf{j}$

(e) $\mathbf{F} = -4\sin 31°\,\mathbf{i} + 4\cos 31°\,\mathbf{j}$

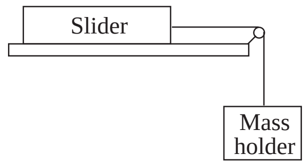
4.4 Friction

In the following activity you will find a mathematical model for the force of friction.

Activity 3 The law of friction

You will need a block to slide on a table, mass holder and masses, small pulley.

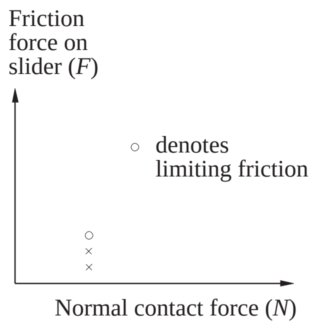
Set up the apparatus as shown opposite.



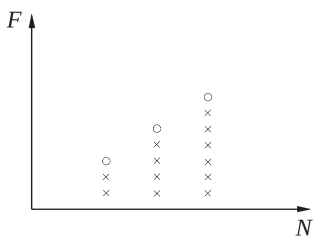
Begin to add 10 g masses to the mass holder, pausing after each mass is added to record the results in a table, as shown below.

total mass of slide	total mass on string	does it slide?
65 g	10 g	No
65 g	20 g	

When the force exerted on the slider by the string attached to the mass holder is **just sufficient** to move the slider, record this point in the table. Under these circumstances the frictional force on the slider, opposing the motion of the slider, is just balanced by the tension in the string. You can assume that the tension in the string is equal to the total weight of the mass holder which is 10 × mass (ie. you can neglect the frictional resistance of the pulley and the elasticity mass of the string). The plane exerts a normal contact force N on the slider, and since the slider is in equilibrium, N is equal to the weight of the slider ($N = mg$). The frictional force increases as the tension in the string increases until it reaches a maximum value, after which increasing the tension in the string will cause the slider to move. This value of the frictional force is called the **limiting** frictional force.



Now draw a graph showing the frictional force on sliding (which equals the tension in the string) against the normal contact force (which equals the total weight of the slider) as shown.



Now increase the mass of the slider by placing, say, 100 g on it and repeat the above experiment, again noting carefully when limiting friction is reached.

Increase the mass of the slider by adding 200 g and then 300 g repeating the above procedure, and you should obtain a graph similar to the one shown opposite.

The points representing limiting friction will lie, approximately, on a straight line passing through the origin, the slope of which you denote by μ. Thus

$$F \le \mu N$$

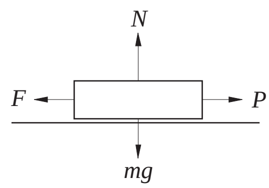
where F is the frictional force on the slider. The dimensionless number μ is called the **coefficient of friction**.

Is there any reason why the coefficient of friction should be less than 1?

Activity 4

Repeat the experiment in Activity 3 attaching sandpaper to the slider and to the plane.

To summarise the law of friction, consider an object of mass m on a horizontal table and suppose that a horizontal force P is applied as shown opposite.



As well as the force P, the other forces acting on the object are the normal contact force, the force of gravity and the force of friction, denoted by N, mg and F respectively. As the force P increases in magnitude the force of friction increases in magnitude until some critical value when the object is on the point of sliding. The law of friction suggests that this critical value is proportional to the magnitude of the normal contact force, so that

$$F = \mu N$$

where μ is a constant called the **coefficient of static friction**. The direction of the force of friction always opposes the direction in which the object is tending to move. Until the critical value is reached

$$F < \mu N.$$

When actually sliding the magnitude of the force of friction is given by

$$F = \mu N .$$

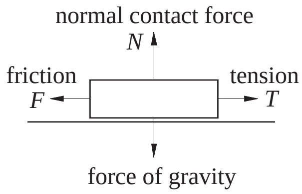
Like the normal contact force, the force of friction is an example of a surface or contact force, applying only when two objects are in contact.

When modelling some physical situations for which the surface of the object and table are sufficiently smooth that the force of friction is negligible, you can make the simplification $F = 0$.

Example

A rope is attached to a crate of mass 70 kg at rest on a flat surface. If the coefficient of friction between the floor and the crate is 0.6, find the maximum force that the rope can exert on the crate before it begins to move.

Solution



The forces acting on the crate are (i) the force of gravity, (ii) the normal contact force, (iii) the friction force and (iv) the tension in the rope.

Since there is no motion of the crate, the upward force (the normal contact force) must balance the downward force (the force of gravity). Hence

$$N = 70g = 700\,\text{N (taking } g = 10) .$$

The law of friction states that:

$$F \le \mu N ,$$

so in this case

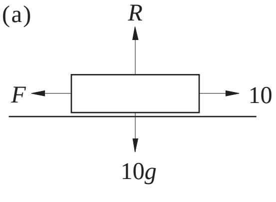
$$F \le 0.6 \times 70g = 420 \text{ N}.$$

As the tension in the rope and the force of friction are the only forces which have horizontal components, the crate will not move unless the tension in the rope is greater than the maximum frictional force. In this case the maximum for the friction is 420 N, and so the maximum force that the rope can exert before the crate begins to move is also 420 N.

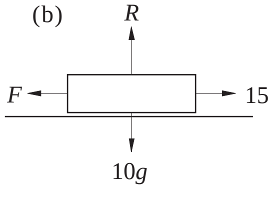
Exercise 4C

1. In the following situations the object is about to slide. Find the coefficient of friction in each case. (In this exercise the normal contact force is labelled R).

(a)



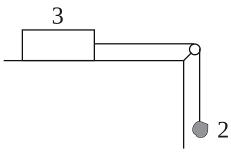
(b)



2. The coefficient of friction between a sledge and a snowy surface is 0.2. The combined mass of a child and the sledge is 45 kg. What force (acting horizontally) would just move the sledge on a level surface?

3. A small object is being pulled across a horizontal surface at a steady speed by a force of 12 N acting parallel to the surface. If the mass of the object is 5 kg determine the coefficient of friction between the object and the surface.

4. A small brick of mass 3 kg is about to slide along a rough horizontal plane being pulled by a stone of mass 2 kg as shown below



The brick and stone are connected by a light inextensible string passing over a pulley.

(a) Find the coefficient of friction between the brick and the plane.

(b) What reaction does the pulley exert?

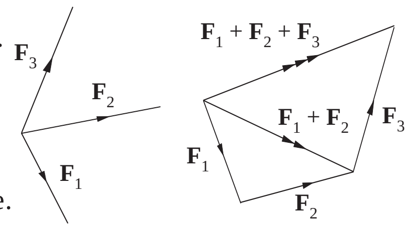
5. A sledge of mass 12 kg is on level ground.

(a) If a horizontal force of 10 N will just move it, find the coefficient of friction μ.

(b) If a girl of mass 25 kg sits on the sledge, find the least horizontal force needed to move it.

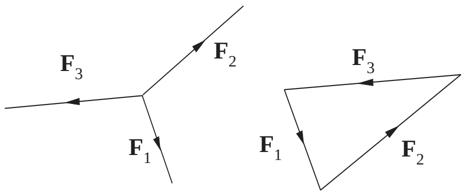
4.5 Forces in equilibrium

When several forces acting on a body produce a change in its motion, then there must be a resultant non-zero force on the body. When there is no change in motion, then the resultant force must be zero and the body is said to be in **equilibrum**.



When three forces act at a point which is not in equilibrum, the non-zero resultant can be found using the parallelogram law twice.

When the forces are in equilibrum, however, the resultant force is zero and the forces form a triangle as Activity 2 shows.



For n forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting at a point in equilibrium,

$$\mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n = \mathbf{0}$$

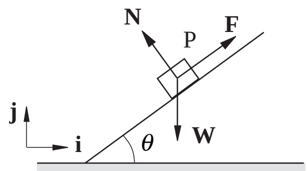
A particle P is at rest on a plane inclined at θ to the horizontal.

The forces acting on P are

(a) its own weight, of magnitude W, acting vertically downwards;

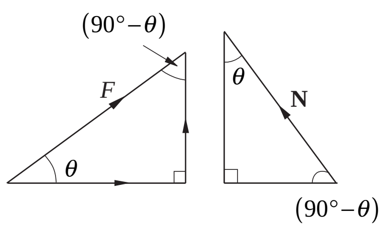
(b) a frictional force, of magnitude F, acting up the plane to prevent P from sliding down it;

(c) a normal contact force, of magnitude N.



To establish a set of equations between these forces a pair of unit vectors need to be chosen. This can be done in several ways but the two most obvious are a pair horizontally and vertically and a pair along and perpendicular to the plane.

(1) Taking the unit vectors horizontally and vertically, as shown, the forces can be written



(i) $\mathbf{W} = -W\mathbf{j}$

(ii) $\mathbf{F} = F\cos\theta\,\mathbf{i} + F\cos(90° - \theta)\,\mathbf{j} = F\cos\theta\,\mathbf{i} + F\sin\theta\,\mathbf{j}$

(iii) $\mathbf{N} = -N\cos(90° - \theta)\,\mathbf{i} + N\cos\theta\,\mathbf{j} = -N\sin\theta\,\mathbf{i} + N\cos\theta\,\mathbf{j}$

Remember that $\cos(90° - \theta) = \sin\theta$.

Since P is in equilibrium

$$\mathbf{W} + \mathbf{F} + \mathbf{N} = \mathbf{0},$$

giving

$$-W\mathbf{j} + (F\cos\theta\,\mathbf{i} + F\sin\theta\,\mathbf{j}) + (-N\sin\theta\,\mathbf{i} + N\cos\theta\,\mathbf{j}) = 0\mathbf{i} + 0\mathbf{j} \text{ so}$$

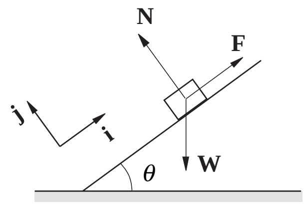
$$(F\cos\theta - N\sin\theta)\mathbf{i} + (-W + F\sin\theta + N\cos\theta)\mathbf{j} = 0\mathbf{i} + 0\mathbf{j}.$$

Equating components gives

$$F\cos\theta - N\sin\theta = 0$$

and

$$-W + F\sin\theta + N\cos\theta = 0$$



(2) Alternatively, taking unit vectors parallel and perpendicular to the plane, the forces can now be written as

(i) $\mathbf{W} = -W\cos(90° - \theta)\mathbf{i} - W\cos\theta\,\mathbf{j} = -W\sin\theta\,\mathbf{i} - W\cos\theta\,\mathbf{j}$

(ii) $\mathbf{F} = F\mathbf{i}$

(iii) $\mathbf{N} = N\mathbf{j}$

Then, since P is in equilibrium,

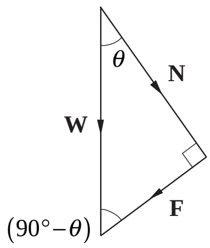
$$\mathbf{W} + \mathbf{F} + \mathbf{N} = \mathbf{0}$$

that is

$$(-W\sin\theta\,\mathbf{i} - W\cos\theta\,\mathbf{j}) + F\mathbf{i} + N\mathbf{j} = 0\mathbf{i} + 0\mathbf{j}$$

so

$$(-W\sin\theta + F)\mathbf{i} + (-W\cos\theta + N)\mathbf{j} = 0\mathbf{i} + 0\mathbf{j}$$



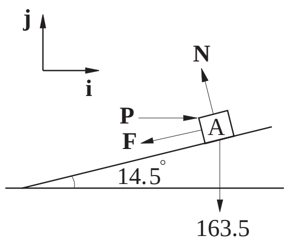
Equating components gives

$$F = W\sin\theta \text{ and } N = W\cos\theta;$$

a much simpler set of equations to work with.

Example

An object, A, lies on a plane inclined at 14.5°. A horizontal force *P*, applied to A, is on the verge of moving A up the plane. The weight of A is 163.5 N, and the coefficient of friction between A and the plane is 0.6. Unit vectors **i** and **j** are defined horizontally and vertically. Show that the normal contact force between A and the plane has a magnitude of about 200 N, and find the magnitude of the force *P*.



Solution

First of all, write each force in component form:

$$\mathbf{N} = -N\cos 75.5°\,\mathbf{i} + N\cos 14.5°\,\mathbf{j}$$

$$\mathbf{P} = P\mathbf{i}, \quad \mathbf{W} = -163.5\,\mathbf{j}$$

$$\mathbf{F} = -F\cos 14.5°\,\mathbf{i} - F\cos 75.5°\,\mathbf{j}$$

Next, since A is stationary, $\mathbf{N} + \mathbf{P} + \mathbf{W} + \mathbf{F} = \mathbf{0}\ (= 0\mathbf{i} + 0\mathbf{j})$.

Equating components gives

$$P - F\cos 14.5° - N\cos 75.5° = 0 \text{ for } \mathbf{i},$$

$$N\cos 14.5° - 163.5 - F\cos 75.5° = 0 \text{ for } \mathbf{j}.$$

Including the equation arising from the law of friction, you have a set of three equations:-

$$P = F\cos 14.5° + N\cos 75.5° \qquad (1)$$

$$N\cos 14.5° - F\cos 75.5° = 163.5 \qquad (2)$$

and $$F = 0.6N.$$

Substituting for F into equation (2) gives

$$N\cos 14.5° - 0.6N\cos 75.5 = 163.5$$

Solving for N,

$$N(\cos 14.5° - 0.6\cos 75.5°) = 163.5$$

$$N = \frac{163.5}{0.8179} = 199.897 \text{ N} \quad (3 \text{ d.p.})$$

and $$N \approx 200 \text{ N}.$$

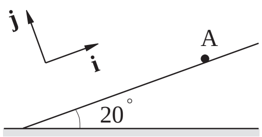
Hence putting $N = 200$ and $F = 0.6\text{ N} = 120$ into equation (1) gives

$$P = 0.6N\cos 14.5° + N\cos 75.5°$$

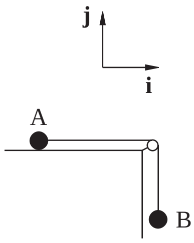
$$= 166.17 \text{ N}.$$

Exercise 4D

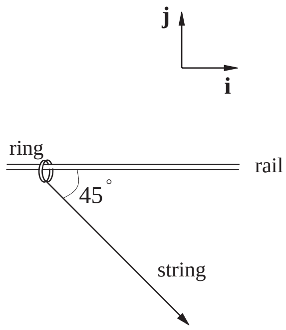
1. A particle A of weight W newtons lies on a plane inclined at $20°$ to the horizontal. A force of P newtons acting directly up the plane is on the verge of pulling A in that direction. Writing each force in **i**, **j**, form, derive the three equations describing the situation.



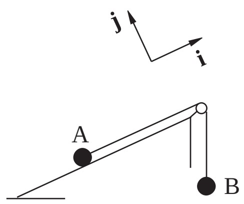
2. Two particles A and B are attached by a light inextensible string passing over a smooth fixed pulley, as shown, and A has twice the mass of B. Find the smallest value of μ, the coefficient of friction, for equilibrium to be maintained.



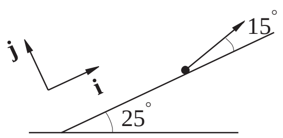
3. A small ring of weight w is threaded on a curtain rail which is lying horizontally. A light inextensible string is attached to the ring and is on the point of pulling it along the rail when the force exerted is equal to $2w$. Find the coefficient of friction between the ring and the rail.



4. A and B are particles connected by a light inextensible string which passes over a smooth fixed pulley attached to a corner of a smooth plane inclined at 37°. Particle B hangs freely. If A has mass 3 kg, find the mass of B given that the system is in equilibrium.



5. A sledge of mass 16 kg is being pulled up the side of a hill of inclination 25°, at a constant velocity. The coefficient of friction between the sledge and the hill is 0.4, and the rope pulling the sledge is at 15° to the hill.

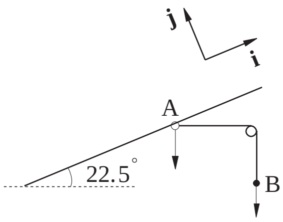


Taking **i** and **j** parallel and perpendicular to the plane, as shown, and modelling the sledge as a particle, write each force acting on the sledge as a vector and find

(a) the magnitude of the normal contact force between the hill and the sledge, and

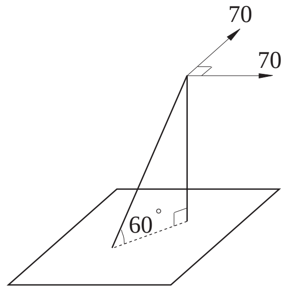
(b) the magnitude of the tension in the rope.

6. A ring A is threaded on a rail which is inclined at $22\frac{1}{2}°$ to the horizontal. It is held in equilibrium on the rail by a string attached to a small weight B, the string passing over a smooth fixed pulley in the plane of the wire.

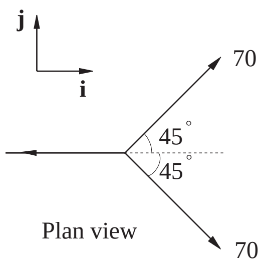


The mass of A is *k* times the mass of B, and the two portions of string are horizontal and vertical respectively. The ring is on the point of moving up the plane, and the coefficient of friction between A and the rail is 0.1. Find the value of *k*.

7. A vertical pole has two horizontal wires attached to it at right-angles to each other. The tension in each wire is 70 N in magnitude. In order to maintain the pole in this position a stay is attached to it and to the ground, the stay making an angle of 60° with the ground.



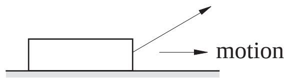
The unit vectors **i** and **j** are in the directions shown on the plan view, while **k** acts vertically upwards.



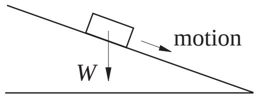
Show that **T**, the tension in the stay, is of magnitude 198 N and hence write **T** as a vector.

4.6 Non-equilibrium situations

In many cases, an object in motion is constrained so that it does not move in the direction of the force causing it to move. You have come across instances of this already, such as



(a) a pulling force operating through a rope on a block on a level surface: although the applied force is inclined from the horizontal, the motion is along the surface;



(b) a block sliding down an inclined plane: the object's weight causes it to move, but clearly not vertically downwards.

There are, of course, other forces acting on the block in each of these cases. The direction of motion is in the direction of the resultant of **all** relevant forces.

Force and acceleration

Up to this point, you have worked with systems of forces in equilibrium. In these cases the resultant force (that is, the sum of all the forces) is zero. In Question 5 of Exercise 4D the 'particle' (a sledge!) is in motion, and you may have been tempted to think that the forces are not then in equilibrium. Remember, though, that Newton's First Law states that bodies will remain at **constant velocity** (possibly zero) until acted upon by an external force. The implication of this is that, as was shown in Chapter 1, change in motion, that is acceleration, implies force and conversely force implies acceleration. Here force means a non-zero resultant force.

$$\text{force} \Rightarrow \text{acceleration, and acceleration} \Rightarrow \text{force}$$

In Chapter 2 you saw that, for a body of constant mass,

$$\text{resultant force} = \text{mass} \times \text{acceleration}$$

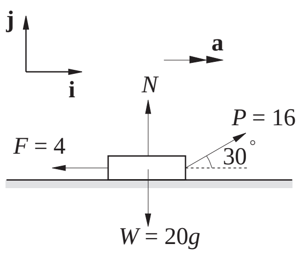
and that a force produces an acceleration in the same direction. Vectorially, then,

$$\boxed{\mathbf{R} = m\mathbf{a}}$$

R is the resultant force acting on the particle which leads to acceleration **a**.

This is Newton's Second Law in vector form.

Example



A block of mass 20 kg is being pulled along level ground by a steady force of 16 N. Friction amounts to 4 N. Calculate the acceleration of the block and the normal contact force between the block and the ground.

Solution

In vector terms,

$$\mathbf{N}+\mathbf{F}+\mathbf{W}+\mathbf{P}=20\mathbf{a}$$

or $\quad N\mathbf{j}-4\mathbf{i}-20g\,\mathbf{j}+\left(16\cos 30°\,\mathbf{i}+16\cos 60°\,\mathbf{j}\right)=20\left(a\mathbf{i}+0\,\mathbf{j}\right).$

Equating components gives

$$(\mathbf{i}) \qquad 16\cos 30°-4=20a \quad \text{so} \quad a=\frac{16\cos 30°-4}{20}$$

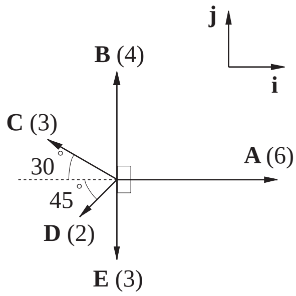
$$=0.493 \text{ ms}^{-2}$$

$$(\mathbf{j}) \qquad N+16\cos 60°-20g=0 \quad \text{so} \quad N=20g-16\cos 60°$$

$$=192\ N$$

Example

The resultant of five forces acting on a particle is **R**. Find the components of each force in the directions indicated and hence **R**. If the mass of the particle is 1.5 kg, find the magnitude and direction of its acceleration under the action of these five forces.



Solution

$$\mathbf{A}=6\mathbf{i},\ \mathbf{B}=4\mathbf{j} \text{ and } \mathbf{E}=-3\mathbf{j}$$

$$\mathbf{C}=-3\cos 30°\,\mathbf{i}+3\cos 60°\,\mathbf{j}$$

$$\mathbf{D}=-2\cos 45°\,\mathbf{i}-2\cos 45°\,\mathbf{j}$$

Thus,

$$\mathbf{R}=\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}+\mathbf{E}$$

$$=\left(6-3\cos 30°-2\cos 45°\right)\mathbf{i}+\left(4-3+3\cos 60°-2\cos 45°\right)\mathbf{j}$$

$$=1.99\,\mathbf{i}+1.09\,\mathbf{j}$$

Using Newton's Second Law:

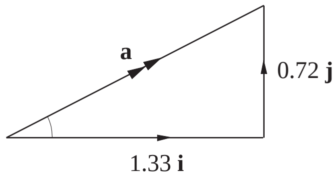
$$\mathbf{R}=1.5\mathbf{a}, \text{ so } \mathbf{a}=\frac{2}{3}\mathbf{R}\approx 1.3\mathbf{i}+0.72\,\mathbf{j}$$

and the acceleration has magnitude

$$\sqrt{1.33^2+0.72^2}=1.51 \text{ ms}^{-2}$$

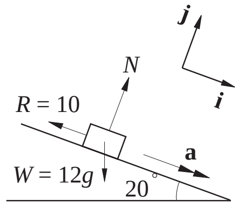
and direction

$$\tan^{-1}\left(\frac{0.72}{1.33}\right) = 28.6° \qquad \text{(anticlockwise from } \mathbf{i}\text{)}$$



Example

A sledge of mass 12 kg slides 40 m down a slope inclined at $20°$. Resistance to motion is 10 N. If the sledge starts from rest, calculate its speed at the bottom of the slope.



Solution

Notice here that it is important to take one of our directions of reference in the same direction as the acceleration. Also, there is no need to consider the **j** components of the forces involved.

Using Newtons's Second Law in the **i** direction gives

$$12g\cos 70° - 10 = 12a$$

so $$a = \frac{12g\cos 70° - 10}{12} = 2.59 \text{ ms}^{-2}$$

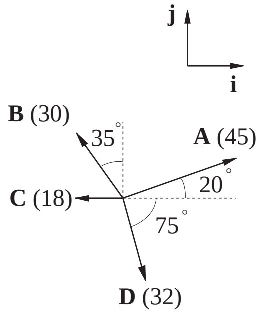
Since a is constant, it is suitable to use the equation $v^2 = u^2 + 2as$ with v the speed at the bottom, $u = 0$ and $s = 40$.

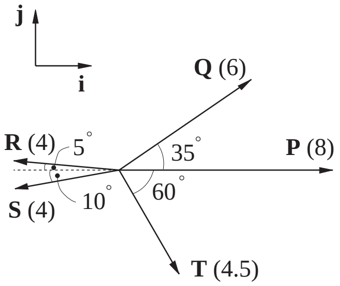
Then $$v^2 = 2 \times 2.59 \times 40 = 207.2$$

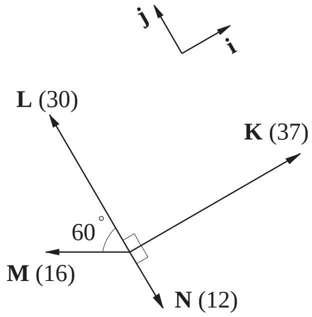
and $$v = 14.4 \text{ ms}^{-1}.$$

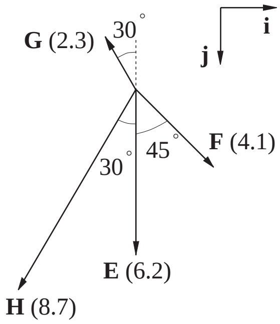
Exercise 4E

1. In each of these four cases, write each force in terms of its components. Hence find the resultant force, **R**, and determine its magnitude and direction.

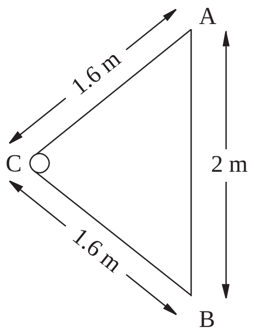






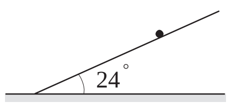


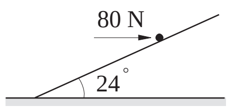
2. A brick of mass 4 kg, resting on a horizontal surface, is acted upon by a force of 20 N pulling at $50°$ to the horizontal. The coefficient of friction between the brick and the surface is $\frac{1}{8}$. Calculate
   (a) the magnitude of the normal contact force on the brick;
   (b) the magnitude of the frictional resistance;
   (c) the speed acquired from rest after 1.5 seconds.
3. An archaeologist investigating the mechanics of large catapults used in sieges of castles demonstrates a simplified plan of such a catapult.



   A rock of mass 20 kg is placed at C. The ropes AC and BC are tensioned to provide acceleration.

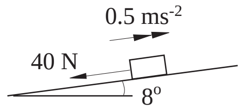
   Measurements suggest an initial acceleration of $30\,\text{ms}^{-2}$ upon release. Calculate the tensions in ropes AC and BC
4. A body of mass 6kg is initially at rest on a slope of $24°$, in limiting equilibrium. Show that μ, the coefficient of friction between the body and the inclined plane, is $\tan 24°$.



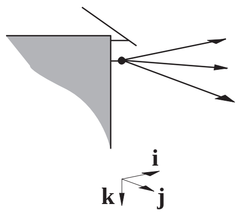


   A horizontal force of 80 N is now applied to the body. Calculate the normal contact force acting on the body and the magnitude of its acceleration up the plane.
5. Find the magnitude of **R**, the resultant of the forces $\mathbf{A} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, $\mathbf{B} = 3\mathbf{j}$, $\mathbf{C} = -2\sqrt{3}\,\mathbf{i} + \sqrt{2}\,\mathbf{k}$ and $\mathbf{D} = -\mathbf{i} + 5\cos 40°\,\mathbf{j} - 5\sin 40°\,\mathbf{k}$.
6. Five forces act on a particle of mass 2 kg. These are $\mathbf{F}_1 = 0.5\mathbf{i} - 0.2\mathbf{j} + 0.1\mathbf{k}$, $\mathbf{F}_2 = 1.7\mathbf{i} + 0.9\mathbf{j}$, $\mathbf{F}_3 = -\mathbf{i} + 0.25\mathbf{k}$, $\mathbf{F}_4 = -0.62\,\mathbf{i} - 0.13\mathbf{j} - \sqrt{2}\,\mathbf{k}$ and $\mathbf{F}_5 = \left(\sqrt{2} - 1\right)\mathbf{i}$.

   Find the accleration of the particle as a vector, and determine its magnitude.
7. A car of mass 800 kg is climbing a hill with an inclination of $8°$. The acceleration of the car is $0.5\,\text{ms}^{-2}$, and resistances to motion are a constant 40 N. Calculate the force exerted by the car.



8. Three communication leads are attached to the side of a building with tensions $\mathbf{T}_1 = 8\mathbf{i} - \mathbf{j} - \mathbf{k}$, $\mathbf{T}_2 = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{T}_3 = 6\mathbf{j} - 5\mathbf{k}$. Find the magnitude of each tension. Find also **R**, the resultant force on the attachment, and its magnitude.



9. Two people are pulling a rock of mass 30 kg across level ground. John pulls with a force of 200 N in a direction of $030°$, while Sheila pulls with a force of 160 N in a direction of $120°$. The coefficient of friction between the rock and ground is 0.6. Assume that the forces exerted on the rock act horizontally.
   (a) Draw a diagram to show the forces.
   (b) Find the resultant of the two pulling forces.
   (c) Find the magnitude of the acceleration of the rock.

4.7 Miscellaneous Exercises

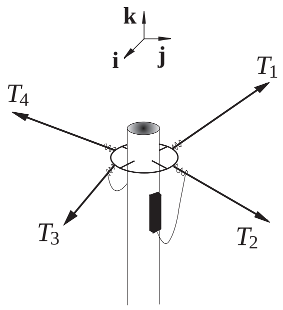
1. (a) Four forces act on the top of a telegraph pole in equilibrium.

$\mathbf{T}_1 = -29\mathbf{i} + 12\mathbf{j} + \mathbf{k}$

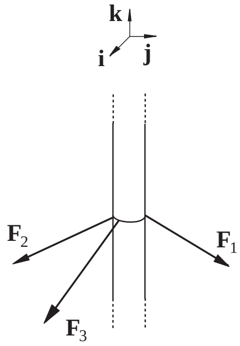
$\mathbf{T}_2 = 8\mathbf{i} + 19\mathbf{j} - 3\mathbf{k}$

$\mathbf{T}_3 = 22\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$

Find $\mathbf{T}_4$.



(b) Three forces, $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ act along stays attached to a point part-way up a vertical flag-pole, which is in equilibrium. Given that $\mathbf{F}_2 = 3\mathbf{i} - 5\mathbf{j} - 15\mathbf{k}$ and $\mathbf{F}_3 = 3\mathbf{i} + 2\mathbf{j} - 18\mathbf{k}$ and the resultant force acts down the pole with magnitude 50 newtons, find $\mathbf{F}_1$.

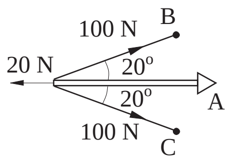


2. A constant force $\mathbf{F} = 16\mathbf{i} - 21\mathbf{j} + 12\mathbf{k}$ is applied to a particle, initially at rest at the origin, and of mass 5.8 kg. Find

(a) the magnitude of its acceleration;

(b) the position vector, **r**, of the particle as a vector function of time.

3. The diagram shows a primitive catapult holding a bolt of mass 600 g. The bolt is attached by strings to points B and C, each string carrying a tension of 100 N. Resistive forces amount to 20 N. Calculate the initial acceleration of the bolt upon release.

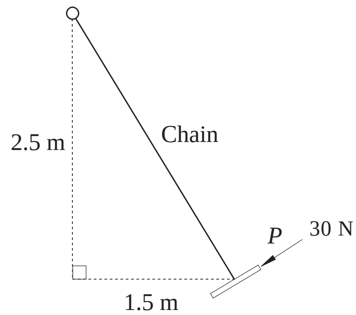


4. A particle of mass 6.5 kg is being pulled up a plane of inclination 36° by a force of 72 N exerted via a piece of string parallel to the plane. The coefficient of friction between the particle and the plane is $\frac{1}{4}$. Find the resultant acceleration of the particle.

5. The diagram shows a simplified model of a swing holding a child with total mass 25 kg. The child is pushed with a force of 30 N in the direction shown. Calculate

(a) the initial acceleration of the child;

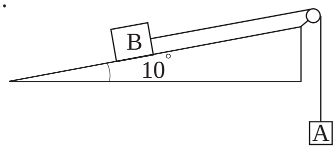
(b) the tension in the chain.



6. Two blocks, A and B, are connected via a light inelastic string which passes over a smooth fixed pulley. A has mass 5 kg and B 3 kg. The coefficient of friction between B and the inclined plane is 0.9. Calculate

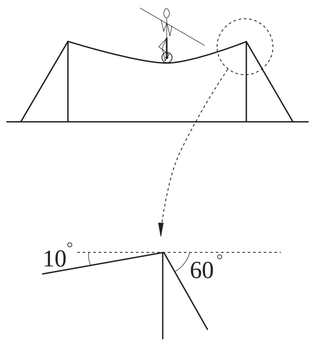
(a) the acceleration of the system;

(b) the distance fallen by A in 2 seconds from rest.



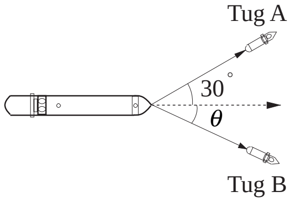
7. A high-wire performer at the circus is riding a uni-cycle across the tightrope. The combined mass of the artiste, the cycle and the balancing pole is 68 kg.

The poles supporting the wire are held by stays attached to the ground at 60° and in the same plane. When the artiste is in the middle of the wire, the wire makes an angle of 10° to the horizontal at either pole.



Find the tension in the tightrope and show that the tension in each stay is approximately 3860 N and, if the mass of each pole is 20 kg, find the force exerted by the ground on one pole.

8. Two tugs are towing a large oil tanker into harbour. Tug A's engines can produce a pulling force of 80000 N while Tug B's engines can produce 65000 N of force.



Defining **i** and **j** suitably, write the forces produced by the tugs in component form, and calculate the angle θ necessary for the tanker to move directly forwards.

Given that there is a resistance to the motion of the tanker of 25 000 N directly opposing motion, find the magnitude of the resultant force on the tanker to the nearest 100 N. Find also the acceleration of the tanker if it has a mass of 20 000 tonnes.

9. Four boys are playing a 'Tug-o'war' game, each pulling horizontally on a rope attached to a light ring. Boy A pulls with a force of $92\mathbf{i} - 33\mathbf{j}$, B with force $66\mathbf{i} + 62\mathbf{j}$ and C with force $-70\mathbf{i} + 99\mathbf{j}$. Given that the ring is in equilibrium, find the force exerted by boy D, and its magnitude.

10. A sledge of mass 30 kg is accelerating down a hill while a boy is trying to prevent it from sliding by pulling on a rope attached to the sledge with a force of 40 N. The hill has inclination 28° and the rope is inclined to the hill at 10°. The coefficient of friction between the sledge and the hill is 0.35.

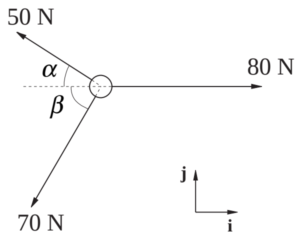
Taking unit vectors **i** down the hill and **j** perpendicular to the hill, write each force in component form. Hence find

(a) the magnitude of the normal contact force on the sledge;

(b) the resultant force on the sledge acting down the hill;

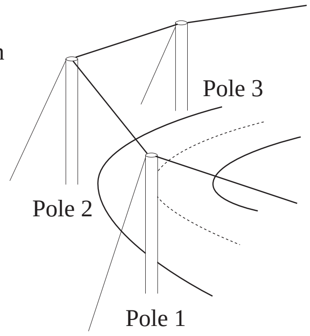
(c) the magnitude of the sledge's acceleration.

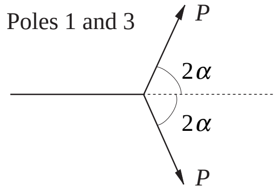
11. Forces act on a telegraph pole along three horizontal wires attached at its top, of magnitudes 80 N, 50 N and 70 N. Taking **i** and **j** parallel and perpendicular to the 80 N force, write each force as a vector. Given that the pole is in equilibrium, determine the values of α and β, the angles that the 50 N and 70 N forces make with the negative **i** direction.

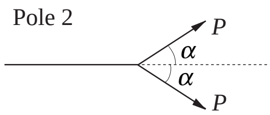


12. A line of (vertical) telephone poles follows the bend of a country road. Each pole has a stay attached to it and to the ground, the stays being symmetrical with respect to the telephone cables connected to the pole, and inclined at θ to the ground. The tension throughout the cables in the system is constant and of magnitude *P*, and these wires may be assumed to be horizontal.

The tensions in the stays at Pole 1 and Pole 3 are equal and of magnitude *T*, while the tension in the stay at Pole 2 is of magnitude $\frac{9T}{2}$. The angles 4α that the cables make with each other at poles 1 and 3 are twice the corresponding angle at pole 2, 2α, as shown.







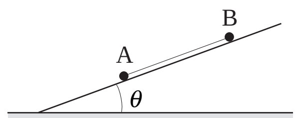
Find the size of the angle α.

13. A particle P of mass 400 g slides down a slope of inclination 32°. Frictional resistance to motion amounts to 1.5 N. Calculate the acceleration of the particle.

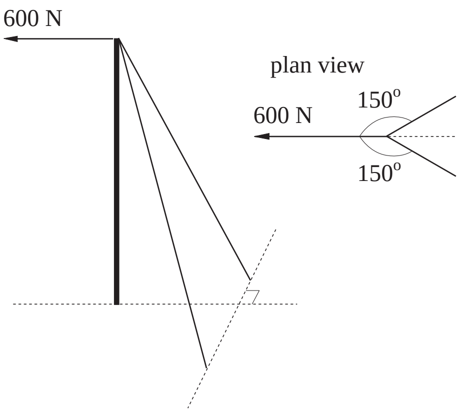
After 4 seconds the particle strikes Q, a second particle of mass 500 g initially at rest. The particles coalesce. Given that the initial speed of P was 10 ms^{-1} calculate the speed of P just before impact and the speed of the coalesced particle just after impact.

14. Two particles on an inclined plane are connected by a light string. A has a mass of 4 kg, B 3 kg, and the coefficient of friction between the plane and the particles is 0.3. Determine *a*, the magnitude of the acceleration of both particles in terms of θ.

When $\theta = 22.3°$ show that $a \approx 1\,\text{ms}^{-2}$ and find the value of θ for which the two particles move at a constant speed.



15. A horizontal wire is attached to the top of a pole, and exerts a force of 600 N. In order to maintain equilibrium, two stays are also attached to the top of the pole, and to the ground at $45°$. The vertical planes containing the stays make angles of 150° with the wire. Assuming the tensions in the stays to be equal in magnitude *T*, show that *T* ≈ 490 N and find the resultant downward force on the pole produced by the stays to the nearest newton.



16. At time *t* seconds a particle P has velocity

$$[(-4\sin 2t)\mathbf{i} + (4\cos 2t - 4)\mathbf{j}]\ \text{ms}^{-1}$$

relative to a fixed origin 0, where **i** and **j** are mutually perpendicular unit vectors.

(a) Find the magnitude of the acceleration of P.

(b) When $t = \frac{\pi}{8}$, find the angle that the acceleration makes with the unit vector **i**.

(c) Given that P passes through 0 at time $t = 0$, find the position vector of P relative to 0 at any subsequent time.

(AEB)

17. At time *t* seconds the position vectors, relative to a fixed origin 0, of two particles P_1 and P_2 are $\mathbf{r}_1$ and $\mathbf{r}_2$ metres respectively, where

$$\mathbf{r}_1 = e^{2t}(-3\mathbf{i} + 5\mathbf{j}),$$
$$\mathbf{r}_2 = (2\cos 3t)\mathbf{i} + (5\sin 3t)\mathbf{j}$$

Find the values of tan 3*t* such that

(a) the particles are moving parallel to each other,

(b) the accelerations of the particles are perpendicular.

(AEB)

18. Three forces $(3\mathbf{i} + 5\mathbf{j})$N, $(4\mathbf{i} + 11\mathbf{j})$N, $(2\mathbf{i} + \mathbf{j})$N act at a point. Given that **i** and **j** are perpendicular unit vectors find

(a) the resultant of the forces in the form $a\mathbf{i} + b\mathbf{j}$,
(b) the magnitude of this resultant,
(c) the cosine of the angle that the resultant makes with the unit vector **i**.

(AEB)

19. Two forces $(3\mathbf{i} + 2\mathbf{j})$N and $(-5\mathbf{i} + \mathbf{j})$N act at a point. Find the magnitude of the resultant of these forces and determine the angle which the resultant makes with the unit vector **i**.

(AEB)

20. Three forces $(\mathbf{i} + \mathbf{j})$N, $(-5\mathbf{i} + 3\mathbf{j})$N and $\lambda\mathbf{i}$N, where **i** and **j** are perpendicular unit vectors, act at a point. Express the resultant in the form $a\mathbf{i} + b\mathbf{j}$ and find its magnitude in terms of λ. Given that the resultant has magnitude 5 N find the two possible values of λ.

Take the larger value of λ and find the tangent of the angle between the resultant and the unit vector **i**.

(AEB)

21. The position vector **r**, relative to a fixed origin 0, of a particle P is given at time *t* seconds by

$$\mathbf{r} = e^{-t}(\cos t\,\mathbf{i} + \sin t\,\mathbf{j})\ \text{m},$$

where **i** and **j** are fixed perpendicular unit vectors. Find similar expressions for the velocity and acceleration of P at any time *t*. Show that the acceleration is always perpendicular to the position vector.

(AEB)

22. The unit vectors **i** and **j** are horizontal and perpendicular to each other. At 11 a.m. on a particular day, two helicopters P and Q are respectively at points A and B at the same height above the earth. Both helicopters are moving, relative to the air, with the same constant speed v kmh^{-1}. Also relative to the air P moves parallel to the unit vector $\frac{3\mathbf{i}}{5} + \frac{4\mathbf{j}}{5}$ and Q moves parallel to the unit vector $\frac{7\mathbf{i}}{25} - \frac{24\mathbf{j}}{25}$. Determine in terms of *v*, **i** and **j** the velocity of P relative to Q.

Given that the position vector of A relative to B is $(4\mathbf{i} + 5\mathbf{j})$ km and that at noon, the position vector of P relative to Q is $(44\mathbf{i} + 225\mathbf{j})$ km, deduce that $v = 125$.

Given also that, at noon, the position vector of P relative to A is $(80\mathbf{i} + 108\mathbf{j})$ km, find, in the form $a\mathbf{i} + b\mathbf{j}$, the constant wind velocity.

Find, to the nearest degree, the angle between the velocities of P and Q relative to the ground.

(AEB)

5 PROJECTILES

Objectives

After studying this chapter you should

- recognise that projectile motion is common;
- understand how to obtain a simple mathematical model of projectile motion;
- be able to validate the model;
- be able to solve simple problems of projectile motion;
- know how to use the model to investigate real life projectile problems.

5.0 Introduction

What do tennis and basket balls have in common with kangaroos?

The ball or body is in motion through the air, the only forces acting on it being its weight and the resistance to its motion due to the air. A motion like this is called a **projectile motion** and is very common especially in sport, for example basketball and tennis. The jumps of insects such as locusts, fleas and grasshoppers are projectile motions, as are the motions of a slate blown off a roof and a piece of mud or small stone thrown up from the road against a car windscreen. Road accidents often involve projectile motions, for example that of the shattered glass of a windscreen. The drops of water that form the jet from a hosepipe behave as projectiles. The Greeks and Romans used catapults to launch projectiles at their enemies, archers were important in medieval battles like Crécy and Agincourt, whilst guns have been a major weapon of war from the sixteenth century onwards.

Activity 1 Projectiles and sport

Make a list of sports which involve projectile motion. How many can you find?

Make a list of non-sporting examples of projectile motion.

Why is it useful to investigate projectile motion?

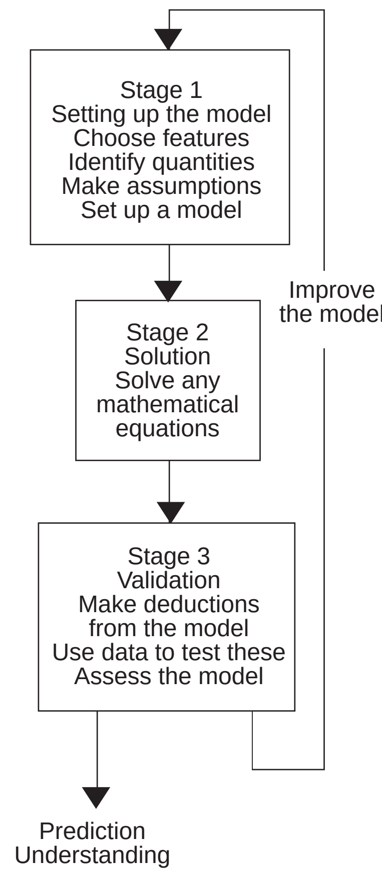
Sports coaches want to know how to improve performance. Police accident investigators want to determine car speeds from the position of glass and other objects at the scene of an accident. In these and other instances mathematical modelling of projectile motion proves very useful.

5.1 Making a mathematical model

You met some examples of mathematical models in Chapter 1, where they take the form of a graph or equation obtained from data collected in experiments. In Chapter 2 you were introduced to Newton's laws; the models in this chapter use these laws to attempt to answer interesting questions about everyday situations. How can I improve my performance at basketball? Why do the chairs swing out on the chair-o-plane ride at a theme park? To shed light on these questions with a mathematical model it is necessary to decide what are the important quantities - e.g. speed, the height of the basket, the weight of the riders in the sport or ride - and to make some assumptions or informed guesses, as to the relations between them.. The resulting model is usually a set of equations. Their solution gives results which need to be tested out against the original situation to see that they make sense. This is called **validating** the model. A model which agrees well with the real situation can be used to make **predictions** about it. One which does not agree well needs modifying, for example, by asking if all the important quantities really have been taken into account.

The diagram opposite summarises this modelling process.

This diagram is basic to what follows. You should use it in any modelling exercises you do.



5.2 Setting up a model for projectile motion

Choosing features and identifying quantities

When a projectile such as a basketball is thrown, it describes a path through the air, first ascending and then descending but also travelling forwards. To describe the motion a mathematical model needs to give the position and velocity of the projectile at any point of its path as functions of time and reproduce the features just described.

Activity 2 What determines the motion of a projectile?

Make a list of the quantities you think determine the motion of a projectile such as a basketball.

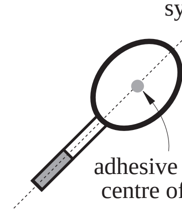
Which do you think are the most important?

You may have on your list the size and shape of the projectile. The next activity explores how important these are.

Activity 3 The motion of a tennis racket

You need a tennis racket (or similar) and a piece of coloured adhesive tape for this activity.

Find the position of the centre of mass of the tennis racket. One way to do this is to find the point about which the racket balances on your finger.



Stick the piece of tape at the centre of mass.

Have two friends throw the racket between them and watch the motion of the tape.

Is the motion of the tape similar to the projectile motion of a ball?

Making assumptions

Activity 3 suggests that, even when the tennis racket rotates, the motion of the centre of mass is as if there were a particle there. This suggests

Assumption 1 The projectile is treated as a particle.

This is reasonable for a tennis or basket ball provided the ball does not have too much spin or rotation. Large amounts of spin on a ball can, however, significantly affect the motion. Treating the ball as a particle leaves out spin and rotation. Even replacing a kangaroo by a particle at its centre of mass gives results which tally with what happens in the hops of real kangaroos.

One way in which spin affects the motion of a ball is by deflecting the motion out of a vertical plane. Side winds can have the same effect. Many projectiles do, however, move in more or less the one vertical plane and these, being the simplest, are the ones modelled here.

Assumption 2 The motion is in one vertical plane.

You may have the resistance of the air to a projectile's motion on your Activity 2 list. However, since many projectiles are in flight for quite short times, air resistance is probably not very important in their motion compared to gravity.

Assumption 3 Air resistance is negligible.

For a model based on these assumptions the motion of the projectile depends at most on

its initial speed U and angle of projection θ;

its point of release;

its mass m;

gravity.

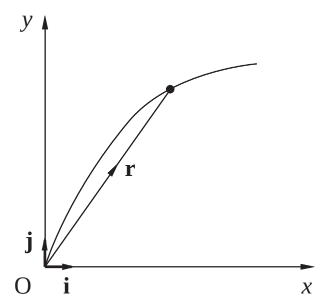
Are these the quantities you decided were most important in Activity 2?

Setting up the model

Projectile motion must obey Newton's Second Law

$$\text{force} = \text{mass} \times \text{acceleration} .$$

Since air resistance is assumed negligible, the only force on the projectile is its weight which acts vertically downward at every point of the path.



To give the position of the projectile at any time t some coordinates are needed. A convenient origin is the point O at which the projectile is released and the time at which this happens is taken as $t = 0$. Cartesian axes Ox, Oy, are then chosen along the horizontal and vertical through O.

The **position** of the projectile at any time t is given by its position vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} ,$$

where **i** and **j** are unit vectors along Ox and Oy. Since **r** varies with t, x and y are functions of t, which it is the aim of the model to determine.

The **velocity v** of the projectile is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j},$$

whilst its **acceleration a** is

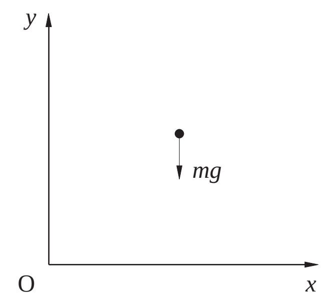
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}.$$

The basic equation of the model

The force on the projectile at time t is

$$\mathbf{F} = 0\mathbf{i} - mg\,\mathbf{j}\,.$$

Newton's Second Law then gives $m\mathbf{a} = \mathbf{F}$ or



$$m\,\frac{d^2\mathbf{r}}{dt^2} = 0\mathbf{i} - mg\,\mathbf{j}.$$

Dividing through by m gives

$$\frac{d^2\mathbf{r}}{dt^2} = 0\mathbf{i} - g\,\mathbf{j}. \qquad (1)$$

This is the basic equation of the model.

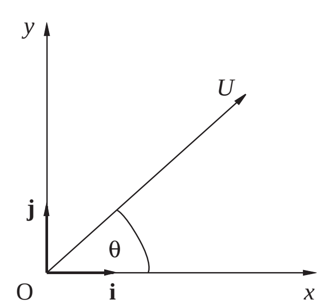
Deduction 1

Since this equation does not involve m, the motion is independent of the mass of the projectile.

5.3 Solving the basic equation of the model

Integrating equation (1) gives

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = A\mathbf{i} + (B - gt)\mathbf{j} \qquad (2)$$



where A and B are constants.

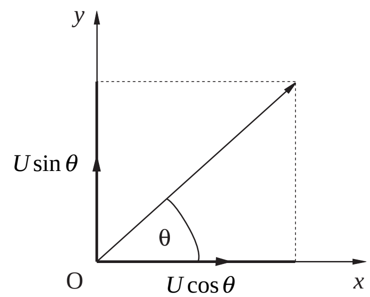
Since the projectile is released from O at time $t = 0$ with speed U at an angle θ to the horizontal, its velocity at $t = 0$ is

$$\mathbf{v} = U\cos\theta\,\mathbf{i} + U\sin\theta\,\mathbf{j}.$$

Using this and substituting $t = 0$ in equation (2) gives

$$U\cos\theta\,\mathbf{i} + U\sin\theta\,\mathbf{j} = A\mathbf{i} + (B - g0)\mathbf{j},$$

so that



$$A = U\cos\theta, \qquad B = U\sin\theta.$$

The velocity at time t is then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = U\cos\theta\,\mathbf{i} + \left(U\sin\theta - gt\right)\mathbf{j}. \qquad (3)$$

Integrating this equation gives

$$\mathbf{r} = \left(Ut\cos\theta + C\right)\mathbf{i} + \left(Ut\sin\theta - \tfrac{1}{2}gt^2 + D\right)\mathbf{j}.$$

where C and D are constants. When $t = 0$, $\mathbf{r} = C\mathbf{i} + D\mathbf{j}$, and, since $\mathbf{r} = 0$ when $t = 0$, $C = D = 0$. This gives

$$\mathbf{r} = Ut\cos\theta\ \mathbf{i} + \left(Ut\sin\theta - \tfrac{1}{2}gt^2\right)\mathbf{j} \qquad (4)$$

Taking components of equations (3) and (4) gives the basic results of the model for projectile motion.

Deduction 1

$$\frac{dx}{dt} = U\cos\theta \qquad (5)$$

$$\frac{dy}{dt} = U\sin\theta - gt \qquad (6)$$

$$x = Ut\cos\theta \qquad (7)$$

$$y = Ut\sin\theta - \tfrac{1}{2}gt^2. \qquad (8)$$

These equations give the position and velocity of the projectile at any time. Do they remind you of any equations you have already met? If so, why do you think this should be?

Equation (5) shows that the horizontal component of velocity, $\frac{dx}{dt}$, is constant throughout the motion.

Could you have predicted this?

Activity 4 A first exploration of the model

You need a calculator for this activity. Take $g = 10$ ms^{-2}.

A tennis ball is given an initial speed of 30 ms^{-1}.

Find the horizontal and vertical components of the velocity of the ball for an angle of projection of $30°$ to the horizontal at times in seconds 0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0. Find also x and y at these times.

At what time is the vertical component of velocity zero? How high has the ball then risen and how far has it travelled horizontally?

When is the ball again on the same level as its point of projection? What is the distance R it has then travelled horizontally and what are its velocity components?

Repeat for angles of projection of $15°$, $45°$, $60°$ and $75°$.

For which of these angles of projection does the ball travel furthest? What do the results you obtain for the distance R for the various angles of projection suggest?

Example

A ball is thrown with a speed of 8 ms^{-1} at an angle of 30° to the horizontal. How high above its point of projection is it when it has travelled 2 m horizontally?

Solution

The time for the ball to travel 2 m horizontally is given by equation (7), $x = Ut\cos\theta$, as

$$t = \frac{x}{U\cos\theta},$$

Here $x = 2$, $U = 8$, $\theta = 30$, so that

$$t = \frac{2}{8\cos 30} = 0.29\,\text{s}$$

Taking $g = 10$ ms^{-2},

the height of the ball above the ground at this time t is given by equation (8) as

$$y = U\sin\theta t - \tfrac{1}{2}gt^2$$

$$= 8\sin 30 \times 0.29 - 5\times(0.29)^2$$

$$\approx 0.74 \text{ m}$$

Hence height is 0.74 m.

Exercise 5A

1. A ball is thrown with initial speed 20 ms^{-1} at an angle of $60°$ to the horizontal. How high does it rise? How far has it then travelled horizontally?

2. A ball is kicked with speed 25 ms^{-1} at an angle of projection of $45°$. How high above the ground is it when it has travelled 10 m horizontally?

3. An arrow is fired from a bow with a speed of $50\,\text{ms}^{-1}$ at an angle of $5°$ to the horizontal. What is its speed and the angle its velocity makes with the horizontal after 0.6 s?

4. A stone is thrown with speed 10 ms^{-1} at an angle of projection of $30°$ from the top of a cliff and hits the sea 2.5 s later. How high is the cliff? How far from the base of the cliff does the stone hit the water?

5.4 Validating the model

Activity 4 suggests that the model is reproducing at least some of the main features of projectile motion in a special case. The projectile ascends then descends and at the same time travels forward. A more testing validation is to deduce what path the model says the projectile follows and then test this experimentally.

The path of the projectile

Activity 5 Plotting the path

You will need graph paper and a calculator or a graphic calculator for this activity.

(a) For $U = 10\text{ ms}^{-1}$, $\theta = 30°$, find the values of x and y at time intervals of 0.1 s from equations (7) and (8).

(b) Plot the points (x, y) to obtain the path of the projectile.

(c) Repeat for $U = 10\text{ ms}^{-1}$, $\theta = 60°$.

You should find that both paths are symmetrical about the vertical through their highest points.

You will need your plot again in Activity 9.

The path of a projectile is an example of a curve called a **parabola**.

The equation of the path for general values of U and θ is found by eliminating the time t between equations (7), $x = Ut\cos\theta$ and (8), $y = Ut\sin\theta - \frac{1}{2}gt^2$.

From equation (7)

$$t = \frac{x}{U\cos\theta}.$$

Substituting this into equation (8) gives

$$y = U\sin\theta\left(\frac{x}{U\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{U\cos\theta}\right)^2$$

Remember $\sec\theta = \dfrac{1}{\cos\theta}$

or

$$y = x\tan\theta - g\frac{\sec^2\theta}{2U^2}x^2$$

Deduction 2

The path of the projectile is the parabola

$$y = x\tan\theta - g\frac{\sec^2\theta}{2U^2}x^2. \qquad (9)$$

Example

A jet of water flows from a hosepipe with speed 40 ms^{-1} at an angle of $60°$ to the horizontal. Given that the particles of water travel as projectiles, find the equation of the path of the jet.

Solution

The equation of the path can be found by substituting $U = 40$, $\theta = 60$, in equations (7) and (8) to give

$$x = 40\cos 60°\, t, \quad y = 40\sin 60°\, t - \tfrac{1}{2}gt^2,$$

or $\quad x = 20t, \qquad y = 20\sqrt{3}\,t - 5t^2.$

From the first of these equations $t = \dfrac{x}{20}$.

Substituting this into the second equation gives

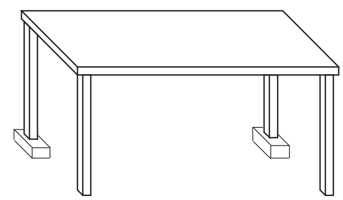
$$y = 20\sqrt{3}\left(\frac{x}{20}\right) - 5\left(\frac{x}{20}\right)^2$$

or

$$y = 1.73x - 0.0125x^2.$$

Testing this deduction experimentally

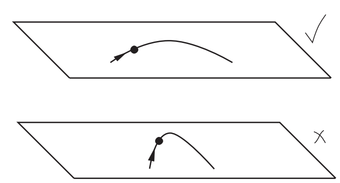
Activity 6 Finding the path of a ball



You need a table, two equal blocks, a billiard or squash ball, a marker pen, sugar paper, Blu-tak, water and squared paper for this activity.

Use the blocks to give the table a gentle incline.

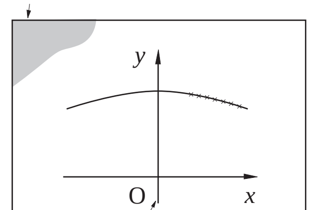
Blu-tack the sugar paper to the table.



Practise releasing the ball so that its path lies well on the paper. A reasonably shallow trajectory as shown in the diagram usually gives the best results.

Wet the ball and release it.

Mark the track before it dries and cut it out.



Put the track on squared paper and insert axes.

Choose some points on the path and write down their coordinates.

Find the equation of the path that goes through these points. You can do this with a graphic calculator or a function graph plotter program.

As it is difficult to plot the path of a ball moving through the air, a table is used in Activity 6.

The acceleration of the rolling ball is down the table but has a value less than *g*. The same model as for motion under gravity can be used for the motion of the ball on the table with the axes O*x* and O*y* along and up the line of greatest slope of the table and *g* replaced by a smaller value than 10 ms^{-2}. In particular, the model predicts the path of the ball is a parabola.

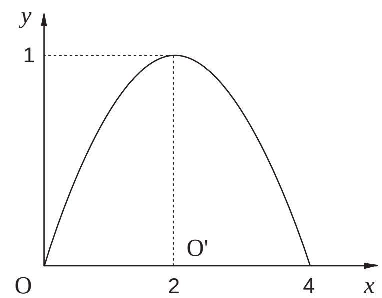
Is this what you find?

The equation you are likely to find is not $y = ax - bx^2$, the form of equation (9). This is because the origin and y-axis are different. It is easier to fit an equation to the path when its axis of symmetry is the y-axis. An example shows how to go from one equation to the other.

Example

For $U = 2\sqrt{10}$ ms^{-1}, and $\theta = 45°$, the path of the projectile from equation (9) is

$$y = -\tfrac{1}{4}x^2 + x\ .$$



Completing the square on the right gives

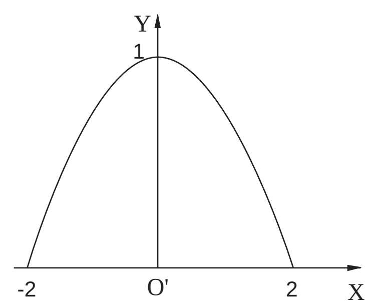
$$y = -\tfrac{1}{4}\left(x^2 - 4x\right)$$

$$= -\tfrac{1}{4}\left[(x-2)^2 - 4\right],$$

so $$y = 1 - \tfrac{1}{4}(x-2)^2.$$

Setting $\mathrm{X} = x - 2, \quad \mathrm{Y} = y$, gives

$$\mathrm{Y} = 1 - \tfrac{1}{4}\mathrm{X}^2$$



referred to the axes shown in the diagram.

You should obtain the equation for the trace of the path in this form, that is, $\mathrm{Y} = c - d\mathrm{X}^2$.

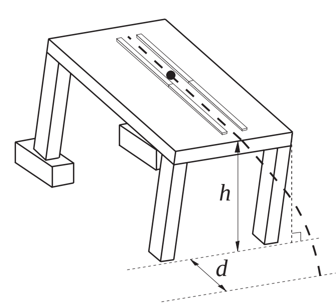
*Activity 7 Finding the path of a ball from photographs

If it is possible to obtain time lapse photos of the motion of a ball, then these could be made the basis of the validation of the equation of the path but now with the motion under gravity.

Activity 8 Another validation you may like to try

You will need a table, two equal blocks, four metre rulers, a squash ball, Blu-tack, a marker pen and paper for this activity.

Use the blocks to give the table a gentle incline longways.



Make a track along the table with the metre rulers as in Galileo's experiment in Chapter 2.

Again, as in Galileo's experiment, find the speed of the ball at the end of the track.

Record this speed.

Find where the ball lands on the floor.

Model the ball as a projectile and from measurements of *h* and *d* calculate its speed of projection on leaving the track.

How does this compare with the speed you found at the end of the track?

The velocity of the projectile

The speed of the projectile at time t is the magnitude $|\mathbf{v}|$ of its velocity **v**, given by

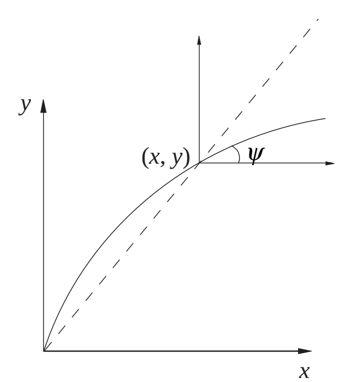
$$|\mathbf{v}| = \left|\frac{d\mathbf{r}}{dt}\right| = \left|\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right|$$

and, since $|a\mathbf{i} + b\mathbf{j}| = \sqrt{a^2 + b^2}$, this can be written as

$$|\mathbf{v}| = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{1}{2}}.$$

The angle ψ the velocity makes with the horizontal is given by

$$\tan\psi = \left(\frac{dy}{dt}\right)\bigg/\left(\frac{dx}{dt}\right).$$



Activity 9 The direction of the projectile's velocity

For $U = 10 \text{ ms}^{-1}$, $\theta = 30°$, find the values of the velocity components $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at time intervals of 0.5 s from equations (5) and (6).

Calculate the angle ψ the velocity makes with the horizontal at the points corresponding to these times.

On the plot of the path you drew in Activity 5 indicate the direction of the velocity at each of these points.

You should confirm that the velocity is along the tangent to the path.

Do you think that your results from Activities 6 to 9 validate the projectile motion model? How might the model be improved?

Example

A stunt motorcyclist takes off at a speed of 35 ms^{-1} up a ramp of 30° to the horizontal to clear a river 50 m wide. Does the cyclist succeed in doing this?

Solution

The cyclist clears the river if the horizontal distance travelled is greater than 50 m.

To find this distance it is necessary to first find the time of flight by putting $U = 35,\ \theta = 30°,\ y = 0$, in equation (8) with $g = 10\ \text{ms}^{-2}$.

This gives

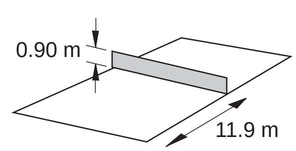
$$0 = 35 \sin 30° \, t - 5t^2 .$$

This equation has two solutions $t = 0$ and $t = 7 \sin 30° = 3.5$ s. The solution $t = 0$ gives the cyclist's take-off time so the time of flight is $t = 3.5$ s.

Substituting $U = 35,\ \theta = 30°,\ t = 3.5$, in equation (7) gives the horizontal distance x travelled as

$$x = 35 \cos 30° \times 3.5 = 106 \text{ m}.$$

So the cyclist easily clears the river.



Example

A tennis player plays a ball with speed 20 ms^{-1} horizontally straight down the court from the backline. What is the least height at which she can play the ball to clear the net? How far behind the net does the ball land when it is played at this height?

Solution

The time for the ball to reach the net is given by equation (7) on substituting $x = 11.90$, $U = 20$, $\theta = 0$, as

$$t = \frac{x}{U\cos\theta} = \frac{11.9}{20} = 0.595 \text{ s}.$$

The distance y below the point of play of the ball after this time is given by equation (8) on substituting $U = 20$, $\theta = 0$, $t = 0.595$ s, $g = 10$ ms^{-2} as

$$\begin{aligned} y &= (20\sin\theta)(0.595) - 5(0.595)^2 \\ &= -1.77. \end{aligned}$$

To clear the net the ball must be played from a height of at least $1.77 + 0.90 = 2.67$ m.

When played from this height, the ball hits the court when $y = -2.67$. The time t when this happens comes from equation (8) by substituting $y = -2.67$, $U = 20$, $\theta = 0$, so that

$$-2.67 = -5t^2$$

which gives $t = 0.73$ s.

From equation (7) the horizontal distance x travelled in this time is

$$x = 20 \times 0.73 = 14.56 \text{ m} \ .$$

The ball then lands $(14.56 - 11.90) = 2.66$ m behind the net.

Exercise 5B

1. David kicks a ball with a speed of 20 ms^{-1} at an angle of $30°$ to the horizontal. How far away from him does the ball land?
2. In the Pony Club gymkhana Carol wants to release a ball to drop into a box. The height above the box from which she drops the ball is 1.5 m and the pony's speed is 12 ms^{-1}. How far from the box should Carol drop the ball?
3. A bowler releases a cricket ball from a height of 2.25 m above the ground so that initially its path is level. Find the speed of delivery if it is to hit the ground a horizontal distance of 16 m from the point of release.
4. Karen is standing 4 m away from a wall which is 2.5 m high. She throws a ball at 10 ms^{-1} at an angle of $40°$ to the horizontal at a height of 1 m above the ground. Will the ball pass over the wall?
5. A bushbaby makes hops with a take-off speed of 6 ms^{-1} and angle of $30°$. How far does it go in each hop?
6. A stone is thrown up at an angle of $30°$ to the horizontal with a speed of 20 ms^{-1} from the edge of a cliff 15 m above sea level so that the stone lands in the sea. Find how long the stone is in the air and how far from the base of the cliff it lands. What are the speed and direction of the stone as it hits the water?
7. A ball is thrown with speed U at an angle of projection of $30°$. Show that at time t its speed q is given by

   $$q^2 = U^2 - Ugt + g^2t^2$$

   Find the height y to which it has risen in this time and hence show that

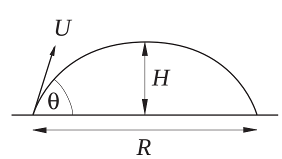
   $$q^2 = U^2 - 2gy \quad .$$

   Do you think this result holds whatever the angle of projection?

5.5 More deductions from the model

Activity 4 explores how high the projectile rises and how far it travels horizontally in particular cases. These questions can, however, be answered generally.

How high does a projectile rise?



A projectile reaches its maximum height H when its vertical velocity component $\frac{dy}{dt} = 0$. The vertical velocity component at time t is given by equation (6)

$$\frac{dy}{dt} = U\sin\theta - gt.$$

Let $\frac{dy}{dt} = 0$ then $0 = U\sin\theta - gt$

which gives

$$t = (U\sin\theta)/g.$$

Putting $y = H$ and $t = \dfrac{U\sin\theta}{g}$ in equation (8) gives

$$H = U\sin\theta.\left(\frac{U\sin\theta}{g}\right) - \frac{1}{2}g\left(\frac{U\sin\theta}{g}\right)^2,$$

so that

$$H = \frac{U^2\sin^2\theta}{2g}.$$

This is the same height as that to which a ball thrown vertically upwards with speed $U\sin\theta$ rises.

Deduction 3

The maximum height the projectile reaches above the point of release is

$$H = \frac{U^2\sin^2\theta}{2g}.$$

How far does a projectile travel horizontally?

The horizontal distance the projectile has travelled when it is again on Ox is called the **range** and is denoted by R.

The total time of flight T is obtained by putting $y = 0$ in equation (8), giving

$$0 = (U\sin\theta)t - \frac{1}{2}gt^2$$

or

$$0 = t\left(U\sin\theta - \frac{1}{2}gt\right).$$

This equation has two solutions, $t = 0$ and $t = \dfrac{2U\sin\theta}{g}$. The solution $t = 0$ is the time of projection from the origin 0 so that $t = \dfrac{2U\sin\theta}{g}$ is the time of flight T.

Deduction 4

The time of flight of the projectile is

$$T = \frac{2U\sin\theta}{g}.$$

This shows the total time of flight is twice the time to the maximum height.

The horizontal distance travelled at time t is $x = Ut\cos\theta$.

From this equation, the range R is

$$R = (U\cos\theta)T = 2U^2\frac{\sin\theta\cos\theta}{g} = \frac{U^2\sin 2\theta}{g}$$

(Remember $\sin 2\theta = 2\sin\theta\cos\theta$)

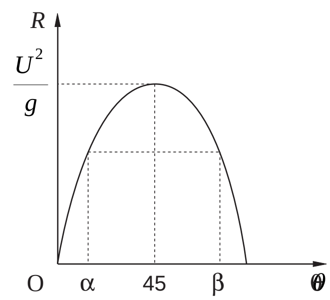
Deduction 5

The range of the projectile is

$$R = \frac{U^2\sin 2\theta}{g}.$$

For a given speed of projection U the range R is a function of θ. It is a maximum, R_{max}, when $\sin 2\theta = 1$, that is $2\theta = 90°$ or $\theta = 45°$.

$$R_{max} = \frac{U^2}{g}$$



graph range R against launch angle θ

The graph of R against θ shows that for values of R less than R_{max} there are two angles of projection which give the same angle. One of these, $\theta = \alpha$, is less than $45°$ and the other, $\theta = \beta$, greater than $45°$. Since the graph is symmetrical about the line $\theta = 45°$,

$$\beta = 90 - \alpha$$

Is this what you found in Activity 4?

Activity 10 Maximum ranges in sports

Some typical initial speeds of projectiles in sports are

golf ball	70 ms^{-1}
long jumper	10 ms^{-1}
rugby and soccer balls	30 ms^{-1}
table tennis ball	25 ms^{-1}
tennis ball	40 ms^{-1}
water polo ball	15 ms^{-1}

Find the maximum range for each of these from the projectile model.

In practice not all the projectiles in Activity 10 attain their maximum range. Why do you think this might be?

Example

A projectile, given an initial speed of 20 ms^{-1}, travels a horizontal distance 30 m. What are its possible angles of projection?

Solution

The time of flight is first found from equation (8); equation (7) then gives the angle of projection θ.

Putting $y = 0$, $U = 20$, in equation (8) gives

$$0 = 20\sin\theta t - 5t^2,$$

which has solutions $t = 0$ and $t = 4\sin\theta$. The time of flight is then $t = 4\sin\theta$. Substituting $x = 30$, $U = 20$, $t = 4\sin\theta$, in the equation for x gives

$$30 = (20\cos\theta)(4\sin\theta)$$

or

$$0.75 = 2\sin\theta\cos\theta$$

so that

$$\sin 2\theta = 0.75$$

and hence $\theta = 24.3°$ or $65.7°$.

Example

A tennis player makes a return at a speed of 15 ms^{-1} and at a height of 3 m to land in the court at a horizontal distance of 12 m from her. What are the possible angles of projection of the ball?

Solution

Let the ball travel a time t before hitting the court, θ the angle of projection.

The horizontal distance, $x = 12$, travelled in this time is given by $x = Ut\cos\theta$ with $U = 15$ as

$$12 = 15t\cos\theta ,$$

so that

$$t = \frac{4}{5\cos\theta}.$$

When the ball hits the court, $y = -3$, so the equation of the path gives

$$-3 = 15\sin\theta\, t - 5t^2.$$

Substituting for t gives

$$-3 = 12\tan\theta - 5\left(\frac{4}{5\cos\theta}\right)^2 ,$$

Remember $\sec\theta = \dfrac{1}{\cos\theta}$

or

$$-3 = 12\tan\theta - \frac{16}{5}\sec^2\theta .$$

This gives

$$\frac{16}{5}\left(1+\tan^2\theta\right) - 12\tan\theta - 3 = 0$$

Remember $\sec^2\theta = 1 + \tan^2\theta$

or

$$\tan^2\theta - \frac{15}{4}\tan\theta + \frac{1}{16} = 0 .$$

Solving this quadratic in $\tan\theta$ gives

$$\tan\theta = \frac{\frac{15}{4} \pm \sqrt{\frac{221}{16}}}{2} = \frac{15 \pm \sqrt{221}}{8}$$

which gives $\theta = 1.0°$ or $75.0°$.

The most likely angle of projection is $1.0°$ since the other angle would give the player's opponent great scope for return.

Exercise 5C

1. A ball is thrown with a speed of 12 ms^{-1} at an angle of $30°$ to the horizontal. Find the maximum height to which it rises, the time of flight and the range. Find also the speed and direction of flight of the ball after 0.5 s and 1.0 s.

2. The initial speed of a projectile is 20 ms^{-1}. Find the two angles of projection which give a range of 30 m and the times of flight for each of these angles. What is the maximum range that can be achieved?

3. A projectile has range 100 m and reaches a maximum height of 20 m. What is its initial speed and angle of projection?

4. What is the least speed of projection with which a projectile can achieve a range of 90 m? What is the time of flight for this speed?

5. Robin Hood shoots an arrow with a speed of 60 ms^{-1} to hit a mark on a tree 60 m from him and at the same level as the arrow is released from. What are his possible angles of projection and which one is he likely to choose?

6. A locust can make long jumps of 0.7 m at a take-off angle of $55°$. Use the projectile model to find its take-off speed and the maximum height it reaches. (The take-off speed of locusts is observed to be about 3.4 ms^{-1}, which is higher than the value found using the projectile model. Why do you think this should be?).

7. A ball is thrown so that it goes as high as it goes forward. At what angle is it thrown?

8. A ball is thrown from a point O with speed 10 ms^{-1} at an angle θ to the horizontal. Show that, if it returns to the ground again at a distance from O greater than 5 m, then θ lies between 15° and 75° whilst the time of flight is between 0.52 s and 1.93 s.

5.6 Using the model

You may like to try some of the following investigations which give you a chance to use the projectile model in some real situations.

Activity 11 Accident!

Police Accident Investigation Units use the projectile model to estimate speeds of vehicles involved in accidents as in this hypothetical case.

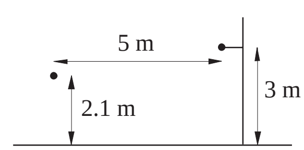
A builder’s van collides with a low stone wall. Several lengths of timber fastened longitudinally to the roof of the van 1.8 m above the ground are projected forwards over the wall to land sticking in a muddy patch of field 9 m in front of the roof. At the same time the windscreen shatters and the bulk of the glass is found in the field beyond the wall between 23 m and 27 m in front of the windscreen.

The Accident Investigation Unit want the likely range of speeds for the car when it struck the wall. Use the projectile model to obtain estimates for this range. You should take into account that these estimates may need to be defended in court.

What might affect the reliability of your estimates?

Activity 12 Making a basketball shot

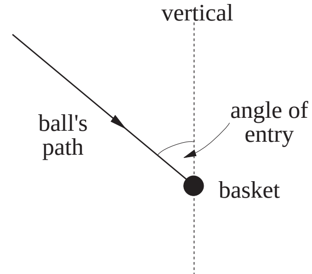
Susan wants to make a clean shot, one that passes through the basket without hitting the rim or bouncing off the backboard. She can give the ball a speed of projection of 8 ms^{-1}.



Treat both the basket and the ball as points.

Investigate the angle of projection that puts the ball into the basket and the corresponding angle of entry.

What angle of projection should Susan aim for?



In reality the basket has a diameter of 0.45 m and the ball a diameter of 0.24 m.

Investigate how much margin Susan has on her preferred angle of projection.

Activity 13 How to improve at shot putting

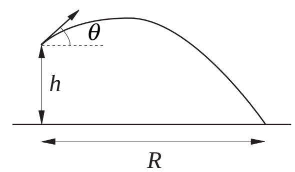
Coaches often concentrate on improving the speed at which a putter projects the shot rather than the angle. Why?

Choose a value for θ, say $\theta = 45°$, the angle for maximum range and a value for *U*, a reasonable one being $U = 12\ \text{ms}^{-1}$. (The height from which the shot is projected is ignored for simplicity)

Investigate the percentage changes in the range *R* for changes in *U*, say 5, 10, 15, 20% increases.

For a given *U*, again, say $U = 12\ \text{ms}^{-1}$, choose values of θ less and greater than $45°$, say $\theta = 42°$ and $\theta = 48°$, and find the percentage changes in *R* for changes in *U* about these values.

What conclusions do you reach?



A putter releases the shot at some height above the ground. The range *R* is measured along the ground.

Investigate numerically how *R* varies with different heights *h* of projection and what angles give the maximum range for different *h*. Typical values of *h* are around 2 m.

Does a tall shot putter have an advantage over a short one?

5.7 Miscellaneous Exercises

1. Rashid is standing 4 m away from a wall which is 5 m high. He throws a ball at 10 ms^{-1} at an elevation of $40°$ above the horizontal and at a height of 1 m above the ground. Will the ball pass over the wall?

   If he throws at an angle θ, show that θ must satisfy

   $$\tan^{-1} 2 < \theta < \tan^{-1} 3$$

   for the ball to clear the wall.

2. A particle is projected with speed *U* at an angle θ to the horizontal. Show that its velocity at the point (x, y) on its path makes an angle ψ with the horizontal, where

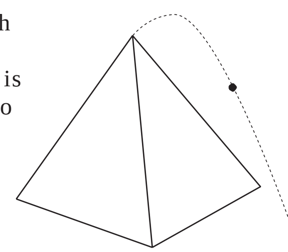
   $$\tan\psi = \frac{2y}{x} - \tan\theta .$$

   For $U = 10\ \text{ms}^{-1}$, $\theta = 45°$, show that

   $$y = x - 0.1x^2 .$$

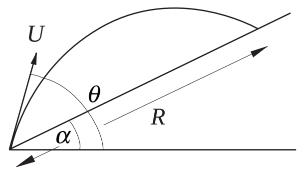
   Hence graph ψ against *x* for *x* at intervals of 0.5 m. What do you deduce about ψ?

3. One of the Egyptian pyramids is 130 m high and the length of each side of its square base is 250 m. Is it possible to throw a stone with initial speed 25 ms^{-1} from the top of the pyramid so that it strikes the ground beyond the base?



4. A ball is projected with speed U at an angle θ to the horizontal up the line of greatest slope of a plane inclined at an angle $\alpha(\alpha < \theta)$ to the horizontal. The ball strikes the plane again at a distance R from the point of projection. Show that

$$R = \frac{2U^2 \sin(\theta - \alpha)\cos\theta}{g\cos^2\alpha}$$

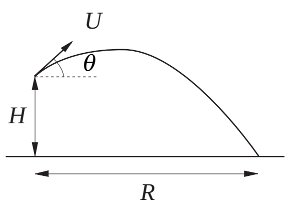


Hence show that the maximum range up the plane is

$$\frac{U^2}{g(1+\sin\alpha)}$$

and is achieved for an angle of projection which bisects the angle between the line of greatest slope of the plane and the vertical through the point of projection.

5. A shot putter can release the shot at a height H above the ground with speed U.



Show that, when the shot is projected at an angle θ to the horizontal, it hits the ground at a horizontal distance R from the point of projection, where R is given by

$$R\tan\theta - g\frac{R^2}{2U^2}\sec^2\theta + H = 0 .$$

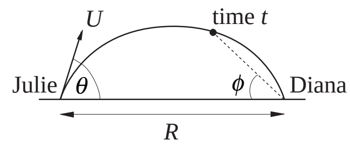
Show that R has a maximum as a function of θ when

$$\tan\theta = \frac{U^2}{gR} .$$

Hence show the maximum range is

$$\frac{U}{g}\left(U^2 + 2gh\right)^{\frac{1}{2}} .$$

6. Julie throws a ball with speed U at an angle θ to the horizontal. Diana, who is at a distance R from Julie, catches the ball at the same height above the ground as Julie throws it.



Show that at any time t the tangent of the angle of elevation of the ball relative to Diana, i.e. the angle ϕ with the horizontal made by the line joining the ball's position at time t to the point where Diana catches it, is given by

$$\frac{gt}{2U\cos\theta}$$

7. A stone is thrown with speed 15 ms^{-1} from a cliff 40 m high to land in the sea at a distance of 30 m from the cliff. Show that there are two possible angles of projection and that they make a right angle.

8. Stones are thrown with speed 15 ms^{-1} to clear a wall of height 5 m at a distance of 15 m from the point of projection. Show that no stone can land behind the wall within a distance of 3 m.

9. A fireman is directing water into a window at a height of 10 m above the ground. Because of the heat he wants to stand as far back from the window as possible. The speed with which the water leaves the hose is 50 ms^{-1}. How far back can he stand?

10. (In this question you should assume g is 9.8 ms^{-2})

A particle P is projected from a point O on level ground with speed 50 ms^{-1} at an angle $\sin^{-1}\left(\frac{7}{25}\right)$ above the horizontal. Find

(a) the height of P at the point where its horizontal displacement from 0 is 120 m,

(b) the speed of P two seconds after projection,

(c) the times after projection at which P is moving at an angle of $\tan^{-1}\left(\frac{1}{4}\right)$ to the ground.

(AEB)

11. (In this question you should assume $g = 9.8 \text{ ms}^{-2}$)

At time $t = 0$ a particle is projected from a point O with speed 49 ms^{-1} and in a direction which makes an acute angle θ with the horizontal plane through O. Find, in terms of θ, an expression for R, the horizontal range of the particle from O.

The particle also reaches a height of 9.8 m above the horizontal plane through O at times t_1 seconds and t_2 seconds. Find, in terms of θ, expressions for t_1 and t_2.

Given that $t_2 - t_1 = \sqrt{17}$ seconds, find θ.

Hence show that $R = \dfrac{245\sqrt{3}}{2}$.

(AEB)

12. A particle P is projected at time $t = 0$ in a vertical plane from a point O with speed u at an angle α above the horizontal. Obtain expressions for the horizontal and vertical components of

(a) the velocity of P at time t;

(b) the displacement, at time t, of P from O.

Given that the particle strikes the horizontal plane through O at time T show that

$$T = \frac{2u\sin\alpha}{g}.$$

Find, in terms of g and T, the maximum height that P rises above the horizontal plane through O.

Given also that, at time $\dfrac{3T}{4}$, the particle is moving at right angles to its initial direction, find $\tan\alpha$.

(AEB)

13. A particle projected from a point O on level ground first strikes the ground again at a distance $4a$ from O after time T. Find the horizontal and vertical components of its initial velocity. (AEB)

14. A particle P is projected from a point O on a horizontal plane with speed v in the direction making an angle α above the horizontal. Assuming that the only force acting is that due to gravity, write down expressions for the horizontal and vertical displacements of P from O at a time t after projection. Given that P lands on the horizontal plane at the point A, show that $\text{OA} = 2v^2 \sin\alpha\cos\alpha / g$.

Find the height above the plane of the highest point B of the path of P.

A second particle Q is projected from O with speed v in the direction OB and lands on the plane at C. Find OC.

Find also the value of $\tan\alpha$ so that A coincides with C.

(AEB)

15. A particle is projected at time $t = 0$ with speed 49 ms^{-1} at an angle α above the horizontal. The horizontal and vertical displacements from 0, the point of projection, at time t are x m and y m respectively. Obtain x and y in terms of α, g and t and hence deduce that, when $x = 140$ and $g = 9.8 \text{ ms}^{-2}$, $y = 140\tan\alpha - 40(1 + \tan^2\alpha)$.

Find the numerical values of the constants a and b so that this equation can be re-written as

$$y = a - 40(\tan\alpha - b)^2.$$

The particle has to pass over a wall 20 m high at $x = 140$, find

(a) the value of $\tan\alpha$ such that the particle has the greatest clearance above the wall,

(b) the two values of $\tan\alpha$ for which the particle just clears the wall.

(AEB)

16. Unit vectors **i** and **j** are defined with **i** horizontal and **j** vertically upwards. At time $t = 0$ a particle P is projected from a fixed origin O with velocity $nu(3\mathbf{i} + 5\mathbf{j})$, where n and u are positive constants. At the same instant a particle Q is projected from the point A, where $\overrightarrow{\text{OA}} = a(16\mathbf{i} + 17\mathbf{j})$ with a being a positive constant, with velocity $u(-4\mathbf{i} + 3\mathbf{j})$.

(a) Find the velocity of P at time t in terms of n, u, g and t. Show also that the velocity of P relative to Q is constant and express it in the form $p\mathbf{i} + q\mathbf{j}$.

(b) Find the value of n such that P and Q collide.

(c) Given that P and Q do not collide and that Q is at its maximum height above A when at a point B, find, in terms of u and g, the horizontal and vertical displacements of B from A.

(AEB)

6 WORK and ENERGY

Objectives

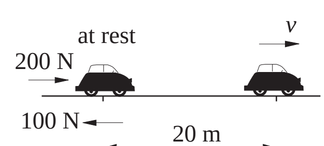
After studying this chapter you should

- be able to calculate work done by a force;
- be able to calculate kinetic energy;
- be able to calculate power;
- be able to use these quantities in solving problems;
- be able to model problems involving elastic strings and springs;
- know when mechanical energy is conserved.

6.0 Introduction

When you walk, run or cycle up a hill or go upstairs you know that it takes more effort than on the level. You may take part in gymnastics or skiing, you may play squash or go down the water chute at the local Leisure Centre, or you may go on rides like the log flume or the Corkscrew at the Amusement Park. All these activities involve the ideas of **work**, **energy** and **power** developed in this chapter.

6.1 Work and kinetic energy



Example

To move a car of mass 1000 kg along a level road you need to apply a force of about 200 N in the direction of motion to overcome the resistance and get going. If this resistance is 100 N and you push it 20 m, what is the speed of the car?

Solution

The net force on the car is $(200-100)\,\mathrm{N} = 100\,\mathrm{N}$, so if a is the constant acceleration, Newton's Second Law gives

$$100 = 1000a \qquad (1)$$

giving

$$a = \frac{100}{1000} = \frac{1}{10}\ \mathrm{ms}^{-2}$$

Since the car is initially at rest, if v is its speed after being pushed 20 m

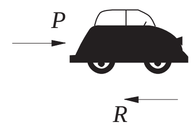
$$v^2 = 0 + 2a \times 20 = 40a = 4$$

or

$$v = 2 \text{ ms}^{-1}.$$

(The result $v^2 = u^2 + 2as$ for constant acceleration was introduced in Chapter 2.)

Consider next the general case of a car of mass m kg. If P is the force given to the car by the person pushing it in the direction of motion, and R the resistance then the equation of motion replacing (1) is



$$P - R = ma$$

so that

$$a = \frac{(P-R)}{m}. \qquad (2)$$

When the car has moved a distance x metres from rest its speed v is given by

$$v^2 = 2ax$$

$$= 2\frac{(P-R)}{m}x, \text{ using equation (2),}$$

or rearranging

$$(P-R)x = \frac{1}{2}mv^2 \ . \qquad (3)$$

The right hand side of equation (3) depends upon the mass and the final speed of the car. It represents the energy of the car due to its motion and is called the **kinetic energy**.

$$\boxed{\begin{aligned}\text{Kinetic energy } &= \frac{1}{2} \times (\text{mass}) \times (\text{speed})^2 \\ &= \frac{1}{2}mv^2\end{aligned}} \qquad (4)$$

The car gains energy as it speeds up. Where has this energy come from? It comes from you pushing the car. To move the car you do work. Some of this work overcomes the resistance and the rest makes the car accelerate. Equation (3) expresses these physical facts mathematically.

Px is called the **work done** by the force P. This is the work you do in pushing the car.

> Work done **by** a constant force
>
> = (Force) $\times$ (distance moved **in the direction of** the force)

The term $-Rx$ in equation (3) is the work done by the resistance. The negative sign occurs since the direction of the force is opposite to the direction of motion.

If the work done by a force is **negative**, work is said to be done **against** the force.

The left hand side of equation (3) is the work done by the resultant force in the direction of motion and is equal to

(work done **by** the force P) $-$ (work done **agains**t the force R)

It is this positive work done which produces the kinetic energy.

This concept of work is a special case of what is meant by the term 'work' in everyday language. To do work in mechanics an object has to move through some distance. The athlete moving the weights vertically upwards does work against the force of gravity. (Gravity is acting downwards, the weights are being moved upwards).

To measure both kinetic energy and the work done by a force we need some units and from equation (3) the units of both these quantities will be the same.

The unit of work is the joule (J), so that 1 J is the work done in moving a force of 1 N through 1 m. Thus

$$1\,\mathrm{J} = 1\,\mathrm{Nm}.$$

For the weights of mass 100 kg being lifted 2 m the work done against gravity is

$$100 \times g \times 2 = 2\,000\,\mathrm{J}$$

taking $g = 10\,\mathrm{ms}^{-2}$.

Since the left hand side of equation (3) represents work done and the right hand side kinetic energy, equation (3) is called the **work-energy equation**.

The jumping flea

The flea Spilopsyllus weighs about 0.4 mg and leaves the ground, when it jumps, at about $1\,\mathrm{ms}^{-1}$. What is its kinetic energy?

In this example the unit of mass needs to be changed to kg. Now

$$0.4\text{ mg} = 0.4 \times 10^{-6}\text{ kg},$$

and

$$\text{kinetic energy} = \frac{1}{2}\left(0.4 \times 10^{-6}\right) \times (1)^2$$

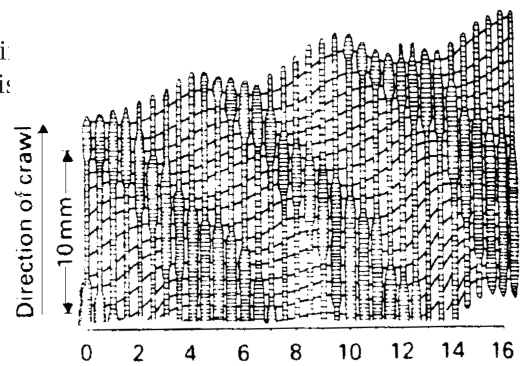
$$= 0.2 \times 10^{-6}$$

$$= 2 \times 10^{-7}\text{ J}.$$

How is this energy used? Can you say anything about how high the flea jumps?

Why does the earthworm move more slowly than the centipede?

Kinetic energy changes are involved here too. When an earthworm, which has no legs, crawls each segment moves in steps coming to a halt between each step. The whole body is continually being given kinetic energy by the muscles doing work and this energy is quickly dissipated at every stage. For this reason the worm only crawls at about 0.5 cms^{-1}. On the other hand, a fast species of centipede only 2.2 cm long can run on its legs at 0.45 ms^{-1}. Although at each step the kinetic energy supplied to its legs fluctuates their mass is small compared to the total mass of the centipede and so the main part of the body, which keeps moving, moves at an almost constant speed and so has almost constant kinetic energy, enabling the centipede to move quickly.



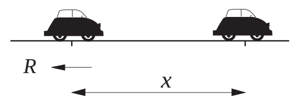
Activity 1 Gravel beds

These are often to be found at the side of some holiday routes and on bendy stretches of road in, for example, the Peak District National Park Their purpose is to bring a vehicle to a halt safely in the event of brake failure or cornering too fast. The car enters the bed at some speed and so possesses some kinetic energy. It is brought to rest without braking by the friction of the bed. As it slows down it loses energy.

What happens to this energy?

Suppose a car of mass 1 tonne enters the bed at 80 kph and stops after 30 m. Find how much kinetic energy is lost and calculate the average resistance force produced by the gravel.

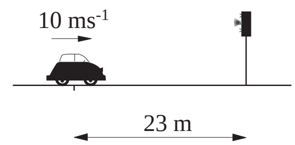
As a simplified model suppose that a car of mass m kg enters the bed with speed v ms^{-1} and comes to rest after x metres. Suppose that the bed is horizontal and that the resistance force, R, is constant.



Find the deceleration in terms of R and m and hence derive the work-energy equation $Rx = \frac{1}{2}mv^2$.

Example Green to amber

A car of mass 1300 kg is travelling along a straight level road at 10 ms^{-1} when the traffic lights change from green to amber. The driver applies the brakes 23 metres from the lights and just manages to stop on the line.



Calculate

(a) the kinetic energy of the car before braking

(b) the work done in bringing the car to rest

(c) the force due to the brakes, friction, etc. against which work is done.

Solution

(a) The kinetic energy of the car before braking

$$= \frac{1}{2} \times 1\,300 \times (10)^2$$

$$= 65\,000 \text{ J}.$$

(b) Using equation (3),

work done in bringing car to rest

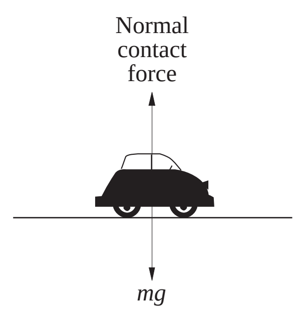
$$= \text{kinetic energy lost}$$

$$= 65\,000 \text{ J}.$$

(c) Since work = (force) × (distance)

$$\text{force} = \frac{\text{work}}{\text{distance}} = \frac{65000}{23} = 2826 \text{ N}.$$

Part (b) of this example illustrates how the work-energy equation can be used to calculate the work done without knowing the forces involved explicitly.



Returning to the model of the gravel bed and also the earlier example of pushing the car, two other forces act on the car

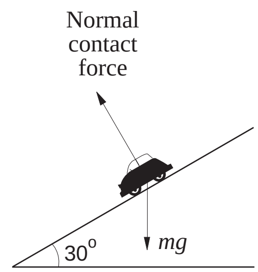
(i) its weight mg,

(ii) the normal contact force.

The distance the car moves in the direction of either of these forces is zero and so these forces do no work.

In general, if the direction of motion is perpendicular to the force, no work is done by the force.

The idea of work can be explored a little further.



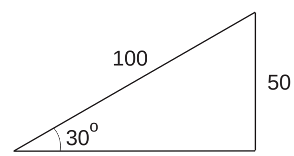
Car on icy slope

The car of mass 1300 kg now slides 100 m down an icy slope, the slope being inclined at 30° to the horizontal. If the frictional forces on the slope and air resistance are ignored the forces acting on the car are

(i) its weight 1300g N

(ii) the normal contact force.

The distance the car moves in the direction of the normal contact force is zero and so this force does no work. The only force which does work is the weight. The distance moved in the direction of the weight is



$$100 \times \sin 30° = 50 \text{ m}$$

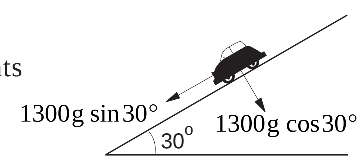
and so

$$\text{work done by gravity} = 1300 \times g \times 50$$

$$= 1300 \times 10 \times 50$$

$$= 650\,000 \text{ J}.$$

In this calculation the distance is found in the direction of the force. Alternatively the weight can be replaced by its components



$1300g \sin 30°$ down the slope

$1300g \cos 30°$ perpendicular to the slope.

The component perpendicular to the slope does no work since it is perpendicular to the motion. The only force which does work is the component down the plane and so

$$\text{the work done} = (1300 \times g \times \sin 30°) \times (\text{distance down the plane})$$

$$= 1300 \times g \times \sin 30° \times 100$$

$$= 650\,000 \text{ J} \quad \text{as before.}$$

In summary,

Work done by a force = (component of force in the direction of motion) × (distance moved).

Activity 2 Pulling a sledge

A girl is pulling a sledge through the snow.

What forces act on the sledge?

Which of these forces does the work?

What happens if the sledge is on a slope?

Exercise 6A

(Take $g = 10\,\text{ms}^{-2}$)

1. Find the kinetic energy of the following:

   (a) a runner of mass 50 kg running at 6 ms^{-1}

   (b) a hovercraft of mass 10^5 kg travelling at 25 ms^{-1}

   (c) an electron of mass 9×10^{-28} g, speed 2×10^{8} ms^{-1}

   (d) a car of mass 1300 kg travelling at 60 kph.

2. A boy of mass 20 kg rides a bicycle of mass 15kg along a road at a speed of 5 ms^{-1}. He applies his brakes and halves his speed; calculate the reduction in the kinetic energy.

3. A bullet of mass 0.02 kg travelling with speed 100 ms^{-1} comes to rest when it has gone 0.4 m into sand. Find the resisting force exerted by the sand.

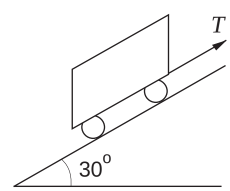
4. A boy of mass 40 kg slides down a chute inclined at $30°$ to the horizontal. If the chute is smooth and the boy starts from rest, with what velocity does he pass a point 5 m from the starting point?

5. A girl of mass 40 kg slides down the water chute 4 m long at the local leisure centre. If the resultant force down the chute is 200 N with what speed does she leave the chute if she starts from rest?

6. If two men push a car of mass 1000 kg, one at each rear corner, and each exerts a force of 120 N at an angle of $20°$ to the direction of motion, calculate the work done by the two men in pushing the car 20 m. Assuming that the resistance is 100 N, calculate the speed of the car if it starts from rest.

7. A car of mass 1200 kg travelling at 15 ms^{-1} comes to rest by braking in a distance of 50 m. If the additional resistance to motion is 100 N calculate the braking force of the car assuming that it is constant.

8. A cable car of mass 1200 kg moves 2 km up a slope inclined at $30°$ to the horizontal at a constant speed. If the resistance to motion is 400 N find the work done by the tension in the cable.



9. A man pushes a lawnmower of mass 30 kg with a force of 30 N at an angle of $35°$ to the horizontal for 5m. The mower starts from rest and there is a resistance to motion of 10 N. Find the work done by the man and the final speed of the mower.

10. A car of mass 1000 kg travelling at 12 ms^{-1} up a slope inclined at $20°$ to the horizontal stops in a distance of 25 m. Determine the frictional force which must be supplied.

11. A 70kg crate is released from rest on a slope inclined at $30°$ to the horizontal. Determine its speed after it slides 10 m down the slope if the coefficient of friction between the crate and the slope is 0.3.

6.2 Work done against gravity: gravitational potential energy

When weights of mass 100 kg are lifted 2 m

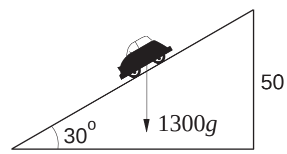
work done **against** gravity $= 100 \times g \times 2$ J

$$= (\text{mass}) \times g \times (\text{vertical height}).$$

Generalising this to the case of weights of mass *m* kg raised vertically through a distance *h* m the work done against gravity is mass $\times$g $\times$vertical height.

mgh is the work done **against** gravity.

In the example of the car sliding **down** the icy slope



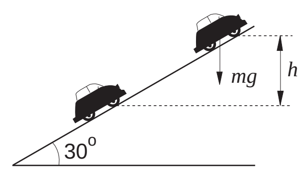
work done **by** gravity $= 1300 \times g \times 50$ J

$= (\text{mass}) \times g \times$ (vertical distance through which the car moved)

If the car had been moving **up** the slope

work done **by** gravity $= -1300 \times g \times 50$ J

$= -(\text{mass}) \times g \times$ (vertical distance through which the car moved)



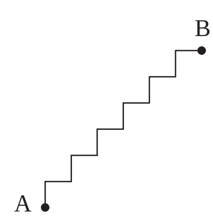
or alternatively,

work done **against** gravity $= 1300 \times g \times 50$ J

$= (\text{mass}) \times g \times$ (vertical distance through which the car moved)

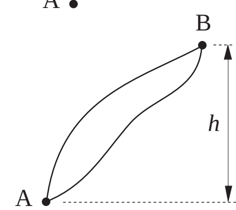
For a car of mass *m* the work done against gravity is still *mgh*. Since this result does not involve the angle of the slope, θ, it is the same for **any** slope.

If you go upstairs is the work done against gravity still your weight times the vertical height?



In fact this result is true for **any** path joining two points A and B. The work against gravity is always

$$mg \times \text{(vertical height from A to B)},$$



and is **independent** of the path joining A to B.

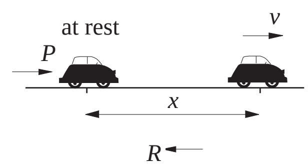
In **raising** a body up through a height h it has **gained** energy due to its change in position. Energy due to position is called **potential energy**. This particular form of potential energy is called **gravitational potential energy**.

Gain in gravitational potential energy in moving an object from A to B = work done against gravity in moving from A to B

= (mass) $\times g \times$ (vertical height from A to B).

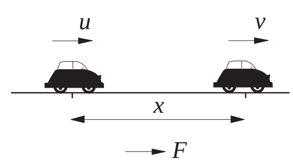
6.3 Work-energy equation. Conservation of energy

For the earlier example of pushing a car the work-energy equation is



$$Fx = \frac{1}{2}mv^2 , \tag{5}$$

where $F = P - R$ is the resultant force in the direction of motion. How is this equation modified if the car moves with constant acceleration, a, from a speed u to speed v over a distance of x metres? If F is the resultant force in the positive x-direction the equation of motion is still



$$F = ma$$

giving

$$a = \frac{F}{m}. \tag{6}$$

From Chapter 2, you know that

$$v^2 = u^2 + 2ax$$

$$= u^2 + 2\frac{F}{m}x, \quad \text{using (6)}$$

or

$$mv^2 = mu^2 + 2Fx$$

Rearranging

$$\boxed{Fx = \frac{1}{2}mv^2 - \frac{1}{2}mu^2} \tag{7}$$

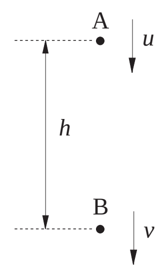
so that

work done by the force F = final kinetic energy − initial kinetic energy

work done by the force F = change in kinetic energy produced by the force.

Equation (7) is the work-energy equation for this situation. It reduces to equation (5) in the special case when $u = 0$.

For a body falling under gravity from A to B, $F = mg$ and using equation (7),



$$mgh = \frac{1}{2}mv^2 - \frac{1}{2}mu^2 \qquad (8)$$

or work done by gravity = change in kinetic energy produced by gravity.

Activity 3   Introducing conservation of energy

In equation (8) the ball falls a height h.

Suppose now you choose a horizontal level so that A is at a height h_1 and B is at a height h_2 above the chosen level.

Write down the work done by the force of gravity as the body falls from A to B in terms of h_1 and h_2.

Show that the sum of the kinetic energy and potential energy is the same at A and at B.

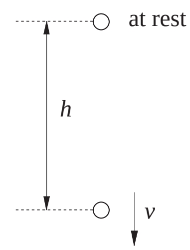
Repeat the calculation with a different horizontal level. What do you find?

Defining

mechanical energy = kinetic energy + potential energy

Activity 3 shows that mechanical energy has the same value at A and B, in other words, mechanical **energy is conserved**.

Activity 3 also shows that it does not matter which horizontal or zero you work from in calculating the potential energy since it is only **changes** in potential energy that are involved. Once you have chosen the zero level, which can often be done to simplify calculations, you must use the same level throughout.



When you release a ball from rest, as the height of the ball decreases its potential energy decreases but the speed of the ball increases and so its kinetic energy increases. If air resistance is neglected the only force acting is gravity and from equation (8) putting $u = 0$

$$mgh = \frac{1}{2}mv^2 \ ,$$

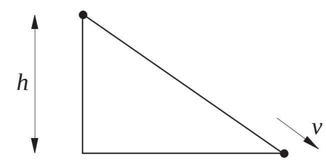
So the decrease in potential energy = increase in kinetic energy. Since m cancels

$$v^2 = 2gh$$

or

$$\boxed{v = \sqrt{2gh}\ .}$$

So, the speed v can be found from considering energy; the speed v is independent of the mass of the ball; it only depends upon the height from which the ball is released.



Does this result hold for a particle sliding down a smooth inclined plane from a vertical height h ?

As the ball falls its potential energy changes to kinetic energy in such a way that at any time the total energy remains the same. If the ball is lifted a vertical height h and then released from rest, the work done against gravity in lifting the ball can be recovered as kinetic energy and for this reason the force of gravity is called a **conservative force**. A property of such forces is that the work done is independent of the path. On the other hand, forces such as friction and air resistance are not conservative because the work done against them is dissipated and so cannot be recovered. It appears mainly in the form of heat.

Activity 4 Dropping a ball from rest

You will need a bouncy ball, 2 metre rulers, and scales to weigh the ball for this activity.

Drop a ball from 2 m above a hard floor surface, releasing the ball from rest.

What happens to the ball? Explain the motion in terms of energy.

Calculate the speed of the ball when it hits the floor.

Why does the ball not bounce back to the same height?

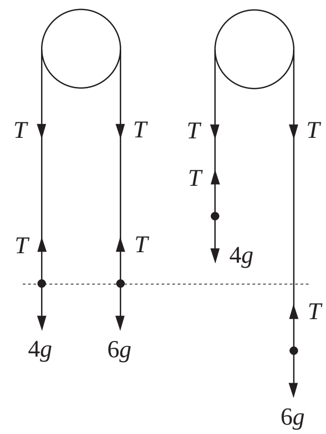
Measure how high the ball bounces.

Calculate the loss of mechanical energy.

If there is a loss of mechanical energy, where does this energy go?

Example

Two masses 4 kg and 6 kg are attached to the ends of an inextensible string over a pulley. The weights of the string and pulley are much smaller than the two masses attached and so are neglected. The pulley is also assumed to be smooth so that the tension T is the same on both sides of the pulley. The masses are held at the same level and the system is released from rest. What is the speed of the 6 kg mass when it has fallen 2 metres?



Solution

Equation (7) can be applied to the two masses separately. For the 4 kg mass the resultant force in the direction of motion is $T-4g$ and so when this mass has risen 2 metres,

$$(T-4g)\times \text{ distance } = \text{ change in kinetic energy}$$

or

$$(T-4g)\times 2 = \frac{1}{2}\times 4v^2 = 2v^2, \qquad (9)$$

where v ms^{-1} is its speed.

For the 6 kg mass the resultant force in the direction of motion is $6g-T$, so that

$$(6g-T)\times 2 = \frac{1}{2}\times 6v^2 = 3v^2, \qquad (10)$$

since the 6 kg mass also has speed v ms^{-1} .

Adding equations (9) and (10) gives

$$12g-8g=5v^2$$

or

$$4g=5v^2$$

giving

$$v^2=\frac{4}{5}g$$

or

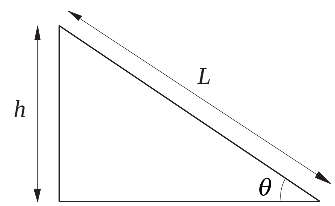
$$v \approx 2.8\,\text{ms}^{-1} .$$

Is it necessary to consider the work done by the tension?
Is mechanical energy conserved for this system?
If so how can the solution be modified?
How would the situation be affected if the pulley was not smooth?
What difference would it make if the rope had a significant weight?

Both this and the falling ball problem illustrate how the 'work-energy' or 'conservation of energy' methods can be used to solve the problem, rather than using Newton's Second Law of motion . This is because, in these examples, you are interested in the speed as a function of distance, and energy methods are quicker in this situation.

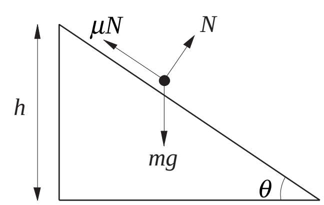
Activity 5 Kinetic and potential energy

Make a list of other activities which involve changes of kinetic and potential energy.



If friction or air resistance are included in the model, mechanical energy is no longer conserved, but the work-energy equation is still applicable.

To illustrate this consider the case of a body moving down a rough inclined plane. An example is a skier coming down a ski slope at speeds when air resistance can be neglected but friction between the skier and the slope is significant. The forces acting on the skier are indicated on the diagram. If the skier starts from rest at the top of the slope and has speed v at the bottom



$$\text{the increase in kinetic energy} = \frac{1}{2}mv^2 \ ,$$

$$\text{distance down the slope} = L = \frac{h}{\sin\theta} \ .$$

Resolving the forces perpendicular to the plane gives

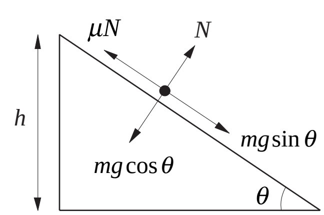
$$N = mg\cos\theta$$

and then the law of friction gives

$$F = \mu\,N = \mu mg\cos\theta.$$

The work done by the weight and friction

$$= (mg\sin\theta)\times L - (\mu mg\cos\theta)\times L$$



$$= mg\sin\theta\left(\frac{h}{\sin\theta}\right) - \mu mg\cos\theta\left(\frac{h}{\sin\theta}\right)$$

$$= mgh - \mu mgh\cot\theta\,.$$

By the work-energy equation

$$\frac{1}{2}mv^2 = mgh - \mu mgh \cot\theta$$

or

$$v^2 = 2gh(1 - \mu \cot\theta) \ .$$

Note that $v^2 > 0$ means $1 > \mu \cot\theta$ or $\tan\theta > \mu$, so the angle of the inclination of the plane, θ, must be sufficiently large for the motion to occur.

If the surface is smooth then $\mu = 0$ and $v^2 = 2gh$ as before. However, for a rough surface $\mu \neq 0$, so that the speed at the bottom of the slope is less than $\sqrt{2gh}$ which is intuitively obvious.

Exercises 6B

1. A man of mass 60 kg climbs a mountain of height 2000 m. What work does he do against gravity?
2. A crane lifts material of mass 500 kg a height 10 m. What is the work done against gravity by the crane?
3. An athlete of mass 80 kg starts from rest and sprints until his speed is 10 ms^{-1}. He then takes off for a high jump and clears the bar with his centre of mass (the point where his weight is assumed to be concentrated) at a height of 2.2 m. How much work has he done up to the moment when he clears the bar?
4. A stone of mass 1 kg is dropped from the top of a building 80 m high .
   (a) Find the total mechanical energy of the stone before it is released and when it has fallen a distance *x* metres and its speed is v ms^{-1}.
   (b) Assuming that energy is conserved show that the speed of the stone *v* and the distance *x* are related by the equation $v^2 = 20x$.
   (c) Find the speed of the stone when it is half-way to the ground and when it hits the ground.
5. A ball of mass 100 g is thrown vertically upwards from a point 2 m above ground level with a speed of 14 ms^{-1}.
   (a) With an origin at ground level, find the total mechanical energy of the ball when it is travelling at speed v ms^{-1} at a height *h* m.
   (b) Assuming that mechanical energy is conserved show that $v^2 + 20h = 236$.
   (c) Calculate the greatest height reached by the ball.
   (d) Calculate the speed with which the ball hits the ground.

6.4 Power

In some situations it is not only the work done which is important but the time taken to do that work. If you are exercising, the rate at which you exercise may be important. When manufacturers state the time taken for a car to reach a certain speed they are saying something about the **rate** at which work can be done by the engine.

The rate at which work is done is called **power**. If it takes 75 s to move a car a distance of 20 m applying a force of 200 N then the average rate of working or the average power is

$$\frac{\text{work done}}{\text{time taken}} = \frac{200\times 20}{75} = 53.3 \text{ N ms}^{-1} = 53.3 \text{ Js}^{-1}.$$

The unit of power is the watt (W) named after the Scottish inventor *James Watt*, famous amongst other things for harnessing the power of steam. 1 watt is defined as 1 joule per second

$$1 \text{ W} = 1 \text{ J s}^{-1}.$$

The power needed to raise 1 kg vertically 10 cm in 1 s is

$$(\text{weight})\times(\text{distance}) = (1\times g)\times\left(10\times 10^{-2}\right) \approx 1 \text{ W}$$

(remembering distances are measured in metres) and so the watt is a fairly small unit. For this reason power is frequently measured in kilowatts (kW). Engineers often quote power in horsepower (h.p.) where

$$1 \text{ h.p.} = 746 \text{ W}.$$

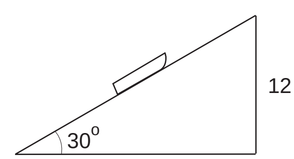
This unit, which is used for cars and lorries, was first introduced by *James Watt* (1736 - 1819). It is based on the rate at which large horses, once used by brewers, worked. It is about ten percent greater than the rate at which most horses can work. The French horsepower is slightly less than the British horsepower.

Activity 6 Modelling the Log Flume at a Theme Park

At the beginning of the Log Flume ride the boat and passengers are carried up the slope by a conveyor belt. In this activity you estimate the power required to get one full boat up the ramp.

Here is the data taken on a recent visit.

Time taken to raise one boat up the slope = 20 s.
(Maximum of 6 persons per boat)
Angle of elevation about 30°.
Height about 12 m.
Mass of boat 150 kg.



Taking the average mass of each passenger as 60 kg, calculate the energy required to raise 1 boat and 6 passengers.

Hence calculate the power required to raise the full boat.

In some cases when the power is specified, the speed is required at a given time. What is the relationship between power and speed? When pushing the car, in the example at the start of Section 6.4, the average power $= 200 \times \frac{20}{75} = (\text{force}) \times (\text{average speed})$.

If a constant force F moves a body in the direction of the force and at a certain time the body is moving with speed v, then what is the power at that time?

Suppose that at time t the distance moved by the body is x metres, then at this time

$$W = \text{work done} = F \times x.$$

Now remembering that power P is rate of doing work

$$P = \frac{d}{dt}(Fx) = F\frac{dx}{dt}\text{, since } F \text{ is constant and } x \text{ varies with } t,$$

$$= Fv,$$

so $$P = Fv \qquad (11)$$

or $$\text{Power} = \text{Force} \times \text{Speed}.$$

Example

A car moves along a horizontal road against a resistance of 400 N. What is the greatest speed (in kph) the car can reach if the engine has a maximum power of 16 kW?

Solution

The car reaches its maximum speed when the engine is at maximum power 16 000 W. The power needed for the car to move against the resistance at speed v ms^{-1} is 400 v, using equation (11). The speed v is a maximum when

$$16000 = 400\, v$$

giving

$$\text{maximum speed} = v = 40 \text{ ms}^{-1}$$

$$= \frac{40}{1000} \times 60 \times 60 \text{ kph}$$

$$= 144 \text{ kph}.$$

Activity 7 How fast does the lorry go up the hill?

The main frictional resistance F on a lorry is due to the air and depends on the velocity v. For a lorry which has a maximum power of 250 kW, a reasonable assumption is $F = 6v^2$.

What is the speed of the lorry going along a level road when it uses half of its power?

If the lorry uses all of its power to go up a hill with slope 1 in 50, how do the maximum speeds compare when the lorry is laden to a total mass of 38 tonnes and 44 tonnes?

You may find it easiest to solve the equations you obtain for the maximum speeds graphically.

Pumping water

Often on roadworks pumps are needed to pump water out of the ground. A pump is required to raise 0.1 m^3 of water a second vertically through 12 m and discharge it with velocity 8 ms^{-1}. What is the minimum power rating of pump required?

Here the work done is in two parts: the water is raised through 12 m and is also given some kinetic energy. Since 1 m^3 of water weighs 1000 kg, the mass of water raised is 100 kg.

The work done in 1 second against this weight of water is

$$(\text{mass raised}) \times g \times (\text{height raised})$$

$$= 100 \times 10 \times 12 = 12000 \text{ J}, \text{ taking } g = 10 \text{ ms}^{-2}.$$

The kinetic energy given to the water in 1 second is

$$= \frac{1}{2} \times (\text{mass raised}) \times (\text{speed})^2$$

$$= \frac{1}{2} \times 100 \times (8)^2 = 3200 \text{ J}.$$

Hence the total work done in 1 second is

$$12000 + 3200 = 15200 \text{ J} .$$

The power required is 15.2 kW.

If the pump is 100% **efficient** this is the rate at which the pump is working to raise the water. In practice some power is lost and pumps work at less than 100% efficiency. Here

$$\text{efficiency} = \frac{\text{power output}}{\text{power input}} \times 100\%$$

If the pump is only 60% efficient, to raise this water it would need to be working at a rate of

$$15200 \times \frac{100}{60} = 25333.\dot{3} \text{ J s}^{-1}$$

$$= 25.\dot{3} \text{ kW}.$$

Since some power is lost, 15.2 kW is the minimum power rating of the pump.

Exercise 6C

1. A crane raises a 5000 kg steel girder at 0.4 ms^{-1}. Assuming that work is not lost in driving the crane, at what rate is the crane working?

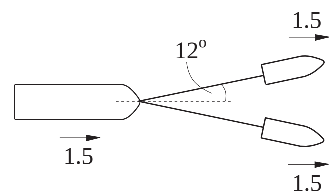
2. A train of mass 5×10^5 kg is travelling at 30 ms^{-1} up a slope of 1 in 100. The frictional resistance is 50 N per 10^3 kg. Find the rate at which the engine is working.

3. A car of mass 1000 kg moves along a horizontal road against a resistance of 400 N and has maximum power of 16 kW. It pulls a trailer of mass 600 kg which is resisted by a force of 300 N. When the speed is 24 kph and the engine is working at maximum power find the acceleration of the car and trailer and the pull in the tow rope.

4. A lorry of mass 5000 kg with an engine capable of developing 24 kW, has a maximum speed of 80 kph on a level road. Calculate the total resistance in newtons at this speed.

   Assuming the resistances to be proportional to the square of the speed, calculate in kilowatts the rate of working of the engine when the truck is climbing a gradient of 1 in 200 at 60 kph and has, at that instant, an acceleration of 0.025 ms^{-2}.

5. Two tugs are pulling a tanker in a harbour at a constant speed of 1.5 knots, as shown. Each cable makes an angle of $12°$ with the direction of motion of the ship and the tension in the cable is 75 kN. Calculate the rate of working of the two tugs. (1 knot = 1.852 kph)



6. Find the power of a firepump which raises water a distance of 4 m and delivers 0.12 m^3 a minute at a speed of 10 ms^{-1}.

7. A pump is required to raise water from a storage tank 4 metres below ground and discharge it at ground level at 8 ms^{-1} through a pipe of cross sectional area 0.12 m^2. Find the power of the pump if it is

   (a) 100% efficient (b) 75% efficient.

6.5 Elasticity – Hooke's Law

So far strings have been assumed to be inextensible. Whilst for some strings this assumption is reasonable, there are other materials whose lengths change when a force is applied.

Activity 8 How do materials behave under loading?

You will need wire, coiled springs, shirring elastic, masses and a support stand for this activity.

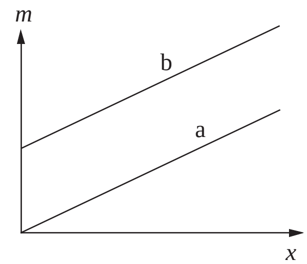
By hanging different masses from the ends of the given materials, investigate the relation between the mass *m*, and the change in the length *x*, from the original length.

Display your results graphically.

Test your specimens to see if they return to their original length when the masses are removed.

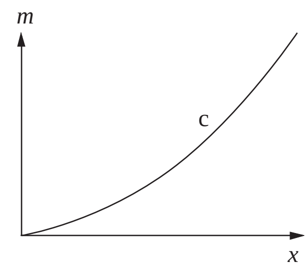
Keep your data for use in Activity 9.

All your specimens are materials which can behave elastically. When you pull the material its length increases but it returns to its original length, sometimes called its **natural length**, when released.

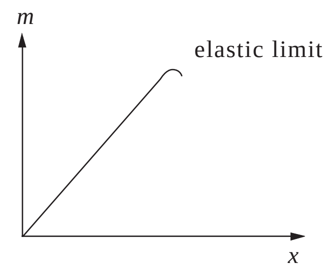


For some of your specimens you may have obtained straight lines such as either (a) or (b), whereas for other specimens a curve like (c) may have occurred. Now

$$\text{Extension} = \text{stretched length} - \text{natural length.}$$



If you add too many masses to the wire it either breaks or keeps extending to the floor. This behaviour occurs when you pass the **elastic limit** and in the case when the wire keeps extending the wire is behaving **plastically**.



When you apply a force to the end of an elastic string and stretch it the string exerts an inward pull or **tension** T which is equal to the applied force. In case (a) if l is the natural length, L the final length and m is the total mass attached then you will have seen from Activity 8 that the extension $x(= L - l)$ and m are linearly related by

$$m = qx \ , \qquad (12)$$

where q is the gradient of the straight line. Although the strings and springs used in Activity 8 have masses, they are very much less than those which are added and so can be neglected.

When a mass attached to the end of a specimen is in equilibrium two forces act, the tension T upwards and the weight mg downwards. Since the mass is at rest

$$T - mg = 0$$

or

$$T = mg. \qquad (13)$$

Eliminating m between equations (12) and (13) gives

$$T = (qg)x \qquad (14)$$

so that T is directly proportional to x. Putting $k = qg$ equation (14) becomes

$$\boxed{T = kx \ .} \qquad (15)$$

This is known as **Hooke's Law**. The constant of proportionality k is called the **stiffness**. This is an experimental result which holds for some materials over a limited range, in some cases only for small extensions. Other materials behave elastically for large extensions but the relation is not linear, e.g. curve (c) above.

Putting $k = \frac{\lambda}{l}$, an alternative form of equation (15) is

$$T = \frac{\lambda}{l}x. \qquad (16)$$

Whereas k depends inversely upon l, the constant λ, called the **modulus of elasticity**, does not depend upon the length but only on the material. Different materials have different values of λ. When $x = l$, $\lambda = T$ and so λ is equal to the tension in the string whose length has been doubled. λ has the units of force and so is measured in newtons. The stiffness $k = \frac{\lambda}{l}$ is measured in Nm^{-1}.

material	modulus of elasticity
aluminium	5.5×10^4
copper	8.8×10^4
brass	7.3×10^4
mild steel	1.6×10^5

Some values of the modulus of elasticity for wires of diameter 1 mm made from different materials

Activity 9 How stiff are the materials?

From your data in Activity 8, where appropriate, determine the values of k and λ for the specimens.

Example

An elastic string of natural length 60 cm has one end fixed and a 0.6 kg mass attached to the other end. The system hangs vertically in equilibrium. Find the stiffness of the string if the total length of the string in the equilibrium position is 72 cm.

Solution

The forces acting on the mass are the tension T upwards and its weight mg downwards. Since the mass is at rest

$$T - 0.6g = 0$$

or

$$T = 0.6g = 6 \text{ N}.$$

The extension of the string $= 12 \text{ cm} = 0.12 \text{ m}$ and so using Hooke's Law

$$T = k(0.12).$$

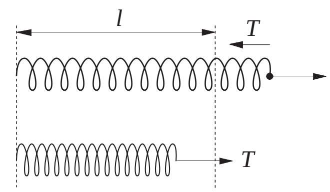
Equating the expressions for T

$$0.12k = 6$$

or

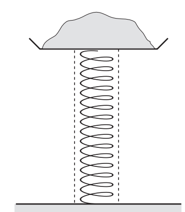
$$k = \frac{6}{0.12} = 50 \text{ Nm}^{-1}.$$

An elastic string is in tension when it is extended. When it is slack its tension is zero. An open-coiled elastic spring can either be extended or compressed. When it is extended there is a tension in the spring, when it is compressed smaller than its natural length there is an outward push, or **thrust**. In both cases these forces tend to restore the spring to its natural length.



Example

A set of kitchen scales consists of a scale pan supported on a spring as shown. For a particular make of scales the stiffness is 2000 Nm^{-1}. Find the compression of the spring when measuring 1.5 kg of flour.



Solution

Modelling the flour as a particle, the forces on the flour are its weight 1.5 g downwards and the thrust T exerted by the spring upwards. Since the system is in equilibrium

$$T - 1.5g = 0$$

or

$$T = 1.5g = 15\text{ N}.$$

Applying Hooke's Law, since the stiffness k is 2000 Nm^{-1} the compression of the spring is given by

$$\text{compression} = \frac{T}{k} = \frac{15}{2000} = 0.0075\text{ m} = 0.75\text{ cm} .$$

Elastic materials do not only occur in the form of strings or springs.

Identify as many situations as possible where elastic materials arise.

Elastic materials also occur in animals and humans. The elastic properties of tendons reduce the work muscles do in hopping or running. In the necks of hoofed mammals there is a highly extensible ligament (ligamentium nuchae) which helps support the weight of the head and serves as a tension spring. Its elastic properties are similar to the load against extension curve (c) at the begining of this section, i.e. they are not linearly related.

Exercise 6D

1. An elastic spring is hanging vertically with a mass of 0.25 kg on one end. If the extension of the spring is 40 cm when the system is in equilibrium, find the stiffness of the spring.

2. An elastic spring of stiffness 0.75 N m^{-1} has one end fixed and a 3 gram mass attached to the other end. The system hangs vertically in equilibrium when the spring has length 49 cm. What is the natural length of the spring?

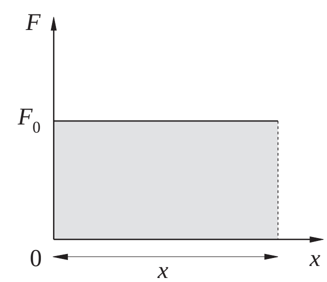
3. A mass of 0.5 kg is attached to the end of an elastic spring hanging vertically with the other end fixed. If the extension of the spring is 8 cm when the system is in equilibrium:

   (a) What is the stiffness of the spring?

   (b) What mass would be required to produce an extension of 10 cm?

6.6 Work done in stretching an elastic string

To extend an elastic string you apply a force. As the extension increases, the tension in the string increases according to Hooke's Law, and so the force you apply increases. You do work against this tension. How much work do you do in extending the string a given distance and what happens to this work? Since the tension varies with distance moved it is necessary to know how to calculate the work done by or against a **variable** force.



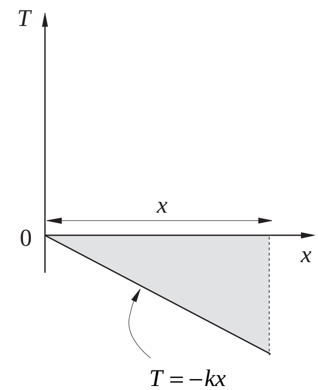
One way of interpreting the work done by the constant force $F = F_0$ **acting in the** x**-direction** is to plot the force against the distance x, then

work done **by** the force $= F_0 \times x$

$=$ shaded area under the straight line.

If the force acts in the direction opposite to x then you can replace F_0 by $-F_0$ and the area is **below** the x-axis indicating that the work done is **against** the force.

This approach can be generalised to the variable tension. In this case the tension T acts against the direction of motion and so T is given by $T = -kx$.



The work done by the tension

$$= -(\text{area of the shaded triangle})$$

$$= -(\frac{1}{2}\times \text{ base } \times \text{ perpendicular height})$$

$$= -\left(\frac{1}{2}\times x \times kx\right)$$

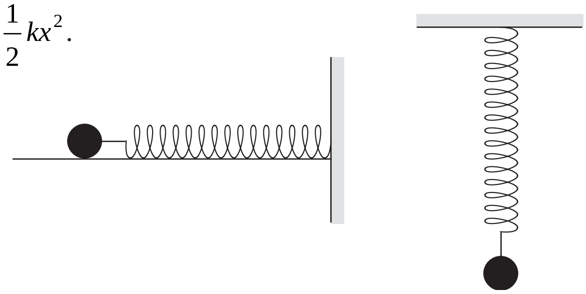
$$= -\frac{1}{2}kx^2$$

and so

$$\boxed{\text{work done } \textbf{against} \text{ the tension } = \frac{1}{2}kx^2} \qquad (17)$$

This is the amount of work you have to do to stretch the string. This energy is stored in the string. Since it depends upon position, it varies with x; it is another type of potential energy and is called **elastic potential energy**. If the string is stretched and then released motion occurs and the amount of work done by the tension as the string returns to its natural length is $\frac{1}{2}kx^2$.

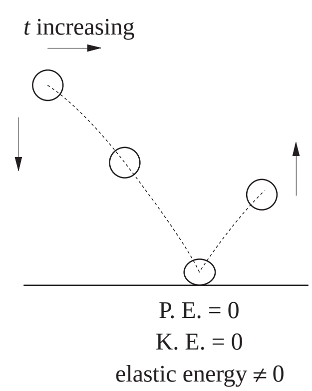
Discuss the motion of a mass attached to the end of a stretched elastic spring either on a horizontal table or suspended vertically.



$$\boxed{\text{Elastic potential energy } = \frac{1}{2}kx^2 = \frac{\lambda x^2}{2l}}$$

This is also the energy stored in an elastic spring which obeys Hooke's Law when it is compressed a distance x. In both the stretched string and compressed spring this energy is recoverable.

Elastic energy is also stored when you squeeze a rubber ball or when a squash ball hits the wall of the court. When an elastic ball lands on the ground it deforms and so has some elastic energy.

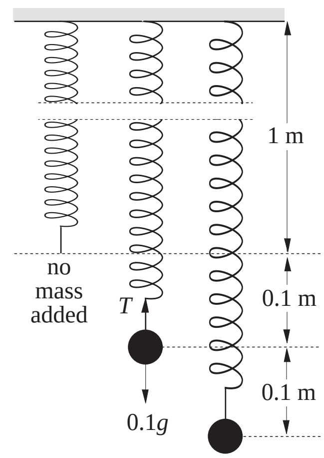


At the point when the ball is instantaneously at rest on the ground (taken to be the zero level of potential energy) all the energy is stored in the ball which then rebounds off the ground at which stage the elastic energy is released and converted into potential energy and kinetic energy.

Elastic energy is stored whenever an elastic material is deformed whatever its shape. For example, when an athlete jumps on a trampoline, the trampoline stores elastic energy which is later released to the athlete.

Example

An elastic string of length 1 m is suspended from a fixed point. When a mass of 100 g is attached to the end, its extension is 10 cm. Calculate the energy stored in the string. When an additional 100 g is attached to the end the extension is doubled. How much work is done in producing this extra 10 cm extension?



Solution

To calculate the energy you need to know the stiffness of the string. Since the 100 g mass is hanging in equilibrium

$$\text{tension upwards} = \text{weight downwards}$$

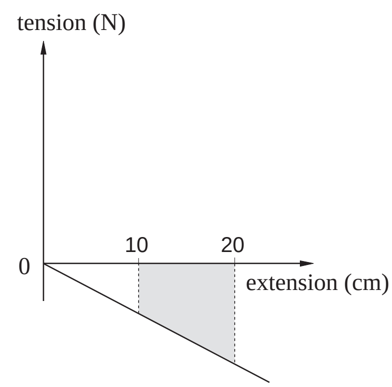
$$k(0.1) = (0.1)g$$

giving $k = g$

and elastic potential energy $= \frac{1}{2}g(0.1)^2 = 0.05$ J.

When an additional 100 g mass is added the string extends a further 10 cm. The extension of the string is now 20 cm and

the elastic potential energy $= \frac{1}{2}g(0.2)^2 = 0.2$ J .



The work done in stretching the extra 10 cm is

$$0.2 - 0.05 = 0.15 \text{ J}$$

which is not the same as the work done in stretching it the first 10 cm.

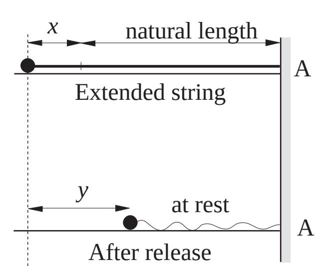
The result is shown in the diagram opposite.

Activity 10 Elastic potential energy

You will need a metre rule, a piece of shirring elastic approximately 1 m in length, Blu-tack and a small mass for this activity.

Attach the mass to one end of the elastic string.

Attach the other end firmly to a table A using Blu-tack.



Extend the string by a known amount x cm ($x = 1, 2, 3, 4, 5$ cm), release the mass and record the distances y cm.

Take at least 6 readings for y for each value of x and average.

Using either a function graph plotter program or graphic calculator find the relationship between x and y.

Write down the work-energy equation to model this situation.

Do your results validate the model?

Exercise 6E

1. An elastic spring has natural length 1.5 m and stiffness 100 Nm^{-1}. Calculate the work done in extending it

   (a) from 1.5 m to 1.7 m,

   (b) from 1.7 m to 1.9 m.

2. An elastic string AB of natural length 3 m and stiffness 4 Nm^{-1} has its end A attached to a fixed point. A force of 4N is applied to the end B. Calculate the work done by the force in extending the string.

3. In a newton metre the distance between readings differing by 1kg is 0.01 m. If the unstretched length of the spring is 0.1 m, find its stiffness. Calculate the energy stored in the spring when a 3 kg mass is attached.

4. A spring whose unstretched length is 0.1 m and stiffness 1000 Nm^{-1} hangs vertically with one end fixed. What extension is produced if a 10 kg mass is suspended at the other end? What additional extension is produced if a further 10 kg mass is added? Calculate the work done against the tension in the spring in producing the additional extension.

6.7 Energy revisited

In Section 6.3, the work-energy equation was formulated in the following form

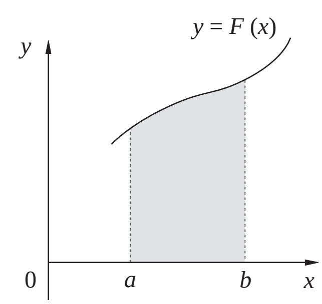
work done by the force = change in kinetic energy produced by the force (18)

For a stretched string the force doing the work varies with distance.

Does equation (18) still hold when the force $F(x)$ varies with the distance x? How do you find the work done by a force which does not depend linearly upon x?

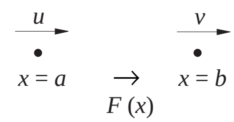
If the curve is not a straight line the work done by the force is given by the area under the graph of $y = F(x)$ from $x = a$ to $x = b$ which is

$$\int_a^b F(x)\,dx .$$



Remember that this integral will be negative if $F(x)$ is negative indicating that the work done is against the force.

If a variable force $F(x)$ moves a body of mass m from $x = a$ to $x = b$ then Newton's law of motion gives



$$F = ma \text{ where } a = \frac{dv}{dt}$$

$$\Rightarrow \qquad F(x) = m\frac{dv}{dt}$$

$$= m\frac{dv}{dx}\,\frac{dx}{dt}$$

$$= mv\frac{dv}{dx}$$

So $$F(x) = \frac{d}{dx}\left(\frac{1}{2}mv^2\right) \qquad \text{(provided } m \text{ is a constant)}$$

since $$\frac{d}{dx}\left(\frac{1}{2}mv^2\right) = m\frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx}$$

$$= mv\frac{dv}{dx}.$$

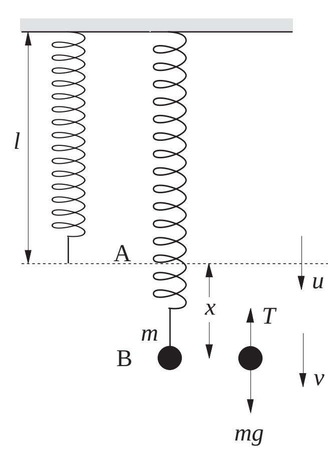
Integrating both sides with respect to x gives

$$\int_a^b F(x)dx = \left[\frac{1}{2}mv^2\right]_u^v = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

where u and v are the velocities at $x = a$ and $x = b$ so that the work done by the force = change in kinetic energy produced by the force.

The work-energy equation does hold in this more general case and it is obtained by integrating the equation of motion.

Mass moving on the end of elastic spring or string



If a 100 g mass is set in motion at the end of an open-coiled elastic spring then as the mass moves up and down the spring which can extend and contract has elastic energy and the mass has both potential energy and kinetic energy.

How are these three forms of energy related?

Consider the question for any mass *m* displaced a distance $l + x$ from the level at which the spring is suspended. Here *l* is the natural length of the spring.

The forces acting on the mass are its weight *mg* downwards and the tension $T = kx$ in the spring upwards.

The resultant force acting on the mass is $(mg - T)$ downwards.

The work done by this resultant force in extending the spring a distance *x* is given by

$$\int_0^x (mg - T)dx = \int_0^x (mg - kx)dx$$
$$= mgx - \tfrac{1}{2}kx^2.$$

Using the work-energy equation (18)

$$mgx - \frac{1}{2}kx^2 = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

or

$$\underbrace{\frac{1}{2}mv^2}_{\substack{\text{kinetic}\\\text{energy}\\\text{at B}}} - \underbrace{mgx}_{\substack{\text{gravitational}\\\text{potential}\\\text{energy at B}}} + \underbrace{\frac{1}{2}kx^2}_{\substack{\text{elastic}\\\text{potential}\\\text{energy at B}}} = \frac{1}{2}mu^2 = \text{constant} \qquad (19)$$

So kinetic energy + gravitational potential energy
+ elastic potential energy = constant (20)

Defining the left hand side of equation (20) as the **mechanical energy** you see that

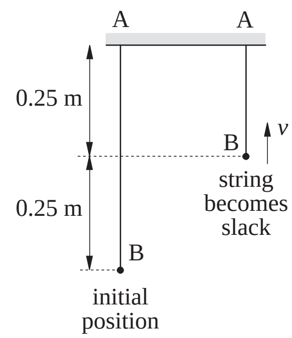
MECHANICAL ENERGY IS CONSERVED.

Example

A 100 g mass is attached to the end B of an elastic string AB with stiffness 16 Nm^{-1} and natural length 0.25 m, the end A being fixed. The mass is pulled down from A until AB is 0.5 m and then released.

Find the velocity of the mass when the string first becomes slack and show that the mass comes to rest when it reaches A.

Solution



Whilst the string is extended it has elastic potential energy. This is zero as soon as the string becomes slack.

Taking the zero level of potential energy to be at the initial position of the mass since the mass is at rest in this position

kinetic energy $= 0$

gravitational potential energy $= 0$

elastic potential energy $= \frac{1}{2}\ k\,(\text{extension})^2$

$$= \frac{1}{2}\ 16(0.25)^2 = 0.5\ \text{J}.$$

Hence total mechanical energy $= 0.5$ J

If v is the speed of the mass when the string becomes slack

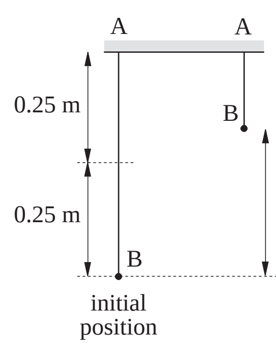
kinetic energy $= \frac{1}{2}(0.1)v^2$

elastic potential energy $= 0$

gravitational potential energy $= (0.1)g\,(0.25) = 0.25$ J.

So total mechanical energy $= 0.05v^2 + 0.25$ J.

If the mass comes to rest a distance h m above the initial position then at this point



kinetic energy $= 0$

elastic potential energy $= 0$

gravitational potential energy $= (0.1)g\,h = h$.

So total mechanical energy $= h$ J.

Since mechanical energy is conserved

$$0.5 = 0.05v^2 + 0.25 = h$$

so that

$$0.05v^2 = 0.5 - 0.25 = 0.25$$

giving

$$v^2 = \frac{0.25}{0.05} = 5$$

or

$$v = 2.2 \text{ ms}^{-1}$$

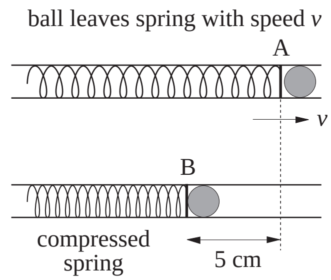
and

$$h = 0.5m \ .$$

When the string becomes slack the mass is travelling at 2.2 ms^{-1}.

Since $h = 0.5\text{m}$ the mass comes to rest at A.

Example



In a horizontal pinball machine the spring is compressed 5 cm. If the mass of the ball is 20 g and the stiffness of the spring is 800 Nm^{-1} what is the speed of the ball when it leaves the spring assuming that friction can be neglected?

Solution

At B the spring is compressed and

$$\text{elastic potential energy } = \frac{1}{2} \times 800 \times (0.05)^2 = 1 \text{ J}$$

$$\text{kinetic energy of ball} = 0 \text{ J}.$$

So $\quad$ total mechanical energy $= 1 \text{ J}$.

At A the spring has returned to its natural length, and

$$\text{elastic potential energy} = 0$$

$$\text{kinetic energy of ball} = \frac{1}{2} \times (0.02) \times v^2$$

$$= 0.01v^2 \text{ J}.$$

So $\quad$ total mechanical energy $= 0.01v^2$ J.

Since energy is conserved

$$0.01v^2 = 1$$

giving

$$v^2 = 100$$

and

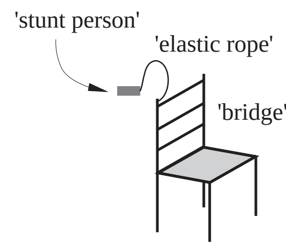
$$v = 10 \text{ ms}^{-1},$$

so that the ball leaves the spring with a velocity of 10 ms^{-1}.

Activity 11 Bungee jumping

You will need shirring elastic, some masses, a metre rule, a table and chair.

You may have seen a film of a stunt where people tie themselves with an elastic rope to a bridge and then jump off. This is known as **bungee jumping**. This activity simulates it safely in the classroom.



Set up the equipment as illustrated in the diagram.

For a given 'person', bridge and rope determine the critical length for the person to survive the jump.

Model the situation mathematically.

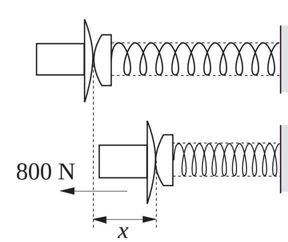
Does your result validate the model?

In the last examples in Section 6.3 when gravity is the only force acting, mechanical energy is conserved. This result is useful in solving this type of problem and leads to quicker solutions than using the equation of motion. The conservation of mechanical energy is used again in the next chapter. Remember that there are situations when mechanical energy is not conserved, in particular when friction and air resistance are included in the models.

The next example shows that the work-energy equation is a more general result which is applicable even when mechanical energy is not conserved.

Example

A 15 tonne wagon travelling at 3.6 ms^{-1} is brought to rest by a buffer (a spring) having a stiffness of $7.5 \times 10^5 \text{Nm}^{-1}$. Assuming that the wagon comes into contact with the buffer smoothly without rebound, calculate the compression of the buffer if there is a constant rolling friction force of 800 N.



Solution

Since friction is included in this problem mechanical energy is **not** conserved. If x is the compression of the buffer when the wagon comes to rest, the forces acting on the wagon are the thrust of the compressed spring and the friction 800 N, both acting in a direction opposite to the direction of motion.

The resultant force is

$$-800-kx=-800-7.5\times10^5 x\,.$$

The work done **by** these forces $=\int_0^x(-7.5\times10^5 x-800)\,dx$

$$=-7.5\times10^5\frac{x^2}{2}-800x\,.$$

The wagon comes to rest and so, remembering that $1\text{ tonne}=10^3\text{kg}$, equation (18) with $v=0$ and $u=3.6\text{ ms}^{-1}$ becomes

$$-7.5\times10^5\frac{x^2}{2}-800x=-\frac{1}{2}15\times10^3\times(3.6)^2$$

or

$$7.5\times10^5x^2+16\times10^2x=15\times10^3\times(3.6)^2$$

which simplifies to

$$750x^2+1.6x-194.4=0\,.$$

This is a quadratic equation whose roots are

$$x=\frac{-1.6\pm\sqrt{(1.6)^2+4\times750\times194.4}}{1500}$$

$$=\frac{-1.6\pm763.677}{1500}$$

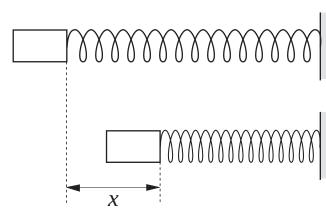
$$=0.508\quad\text{or}\quad-0.510\,.$$

For this problem the negative root has no physical significance and so the required compression is 0.508 m.

Exercise 6F

1. A 100 g mass is attached to the end of an elastic string which hangs vertically with the other end fixed. The string has stiffness 8 Nm^{-1} and natural length 0.25 m. If the mass is pulled downwards until the length of the string is 0.5 m and released, show that the mass comes to rest when the string becomes slack.

2. The mass in Question 1 is replaced by a 75 g mass. Find
   (a) the velocity of the mass when the string first becomes slack;
   (b) the distance below the fixed point where the mass comes to rest again.
3. The 100 g mass in Question 1 is now released from rest at the point where the string is fixed. Find the extension of the string when the mass first comes to rest. Find the speed of the mass when the string becomes slack and show that the mass does not come to rest before reaching the fixed point.
4. A cable is used to suspend a 800 kg safe. If the safe is being lowered at 6 ms^{-1} when the motor controlling the cable suddenly jams, determine the maximum tension in the cable. Neglect the mass of the cable and assume the cable is elastic such that it stretches 20 mm when subjected to a tension of 4 kN.
5. A mass *m* is attached to one end B of an elastic string AB of natural length *l*. The end A of the string is fixed and the mass falls vertically from rest at A. In the subsequent motion, the greatest depth of the mass below A is 3*l*. Calculate the stiffness of the string.
6. A 0.5 kg mass is attached to the end of an elastic string of natural length 2 m and stiffness 0.5 Nm^{-1}. The other end A is fixed to a point on a smooth horizontal plane. The mass is projected from A along the plane at a speed of 4 ms^{-1}. Find the greatest distance from A during the subsequent motion. Show that when the string is taut and the extension is *x* m the velocity *v* of the mass is given by $v^2 = 16 - x^2$.
   By differentiating this equation with respect to *t* show that the acceleration $\frac{d^2x}{dt^2}$ of the mass is given by $\frac{d^2x}{dt^2} = -x$.
   At what distance from A are the maximum values of
   (a) the velocity;
   (b) the deceleration?
7. An energy-absorbing car bumper with its springs initially undeformed has spring stiffness of 525kN m^{-1}. The 1200 kg car approaches a massive wall with a speed of 8 kph. Modelling the car as the mass spring system shown in the diagram, determine
   (a) the velocity of the car during contact with the wall when the spring is compressed a distance *x* m
   (b) the maximum compression of the spring.



6.8 Using scalar products

If a constant force **F** moves through a displacement **d** and the angle between the vectors **F** and **d** is θ, then the work done by **F** is found by multiplying the magnitude of the component of **F** parallel to **d** by the distance moved.

$$\text{Work done by } \mathbf{F} = |\mathbf{F}|\cos\theta|\mathbf{d}|,$$

$$\text{however } |\mathbf{F}||\mathbf{d}|\cos\theta = \mathbf{F}.\mathbf{d}.$$

So, when force **F** is applied to a body while its position vector changes by **d**, the work done is **F.d.**

Example

Find the work done by a force $(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ N which moves a body from the origin to a point P, position vector $(\mathbf{i} + \mathbf{j} + \mathbf{k})$ m.

Solution

Work done $= (2\mathbf{i} + 2\mathbf{j} - \mathbf{k}).(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$$= 2 + 2 - 1$$
$$= 3 \text{ N}.$$

Example

A particle of mass 3 kg is acted upon by three forces
$\mathbf{F}_1 = \mathbf{i} + 2\mathbf{k}$, $\mathbf{F}_2 = 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{F}_3 = 2\mathbf{i} + 3\mathbf{j}$.

If the particle moves from the point $\mathbf{i} - \mathbf{j} - \mathbf{k}$ to $3(\mathbf{i} + \mathbf{j} + \mathbf{k})$, find the work done by the resultant.

Solution

The displacement vector $\mathbf{d} = 3(\mathbf{i} + \mathbf{j} + \mathbf{k}) - (\mathbf{i} - \mathbf{j} - \mathbf{k})$

$$= 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}.$$

Work done by $\mathbf{F}_1 = \mathbf{F}_1.\mathbf{d}$

$$= (\mathbf{i} + 2\mathbf{k}).(2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$$
$$= 2 + 0 + 8 = 10.$$

Work done by $\mathbf{F}_2 = \mathbf{F}_2.\mathbf{d}$

$$= (3\mathbf{j} + 4\mathbf{k}).(2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$$
$$= 0 + 12 + 16$$
$$= 28.$$

Work done by $\mathbf{F}_3 = \mathbf{F}_3.\mathbf{d}$

$$= (2\mathbf{i} + 3\mathbf{j}).(2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$$
$$= 4 + 12 + 0$$
$$= 16.$$

The resultant force $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$

$$= (\mathbf{i} + 2\mathbf{k}) + (3\mathbf{j} + 4\mathbf{k}) + (2\mathbf{i} + 3\mathbf{j})$$
$$= 3\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}.$$

Total work done by $\mathbf{F} = \mathbf{F}.\mathbf{d}$

$$= (3\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}).(2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$$
$$= (3 \times 2) + (6 \times 4) + (6 \times 4)$$
$$= 54.$$

This example shows that, when a set of forces act on a particle, the total work done by the individual forces is equal to the work done by the resultant.

The work done by a variable force **F** can be defined as $\int \mathbf{F}.\mathbf{v}\,dt$ and, since power is the rate of doing work, power is given by **F.v** . Using vector notation means that the scalar product can be used to good effect.

Example

A 5 kg mass moves so that its position vector **r** at time *t* is given by $\mathbf{r} = \sin 2t\,\mathbf{i} + \cos 2t\,\mathbf{j} + 2t\,\mathbf{k}$. Find (a) the kinetic energy and (b) the work done between $t = 0$ and $t = 2$.

Solution

Since $\quad \mathbf{r} = \sin 2t\,\mathbf{i} + \cos 2t\,\mathbf{j} + 2t\,\mathbf{k}$

then $\quad \dfrac{d\mathbf{r}}{dt} = \mathbf{v} = 2\cos t\,\mathbf{i} - 2\sin 2t\,\mathbf{j} + 2\,\mathbf{k}$

and $\quad \dfrac{d\mathbf{v}}{dt} = \mathbf{a} = -4\sin 2t\,\mathbf{i} - 4\cos 2t\,\mathbf{j}$

Hence $\mathbf{F} = m\,\mathbf{a} = 5(-4\sin 2t\,\mathbf{i} - 4\cos 2t\,\mathbf{j})$

(a) Kinetic energy is given by

$$
\begin{aligned}
\text{KE} &= \frac{1}{2}\text{m}\,v^2 \\
&= \frac{1}{2}\times 5\times(2\cos 2t\,\mathbf{i} - 2\sin 2t\,\mathbf{j} + 2\,\mathbf{k}).(2\cos 2t\,\mathbf{i} - 2\sin 2t\,\mathbf{j} + 2\,\mathbf{k}) \\
&= \frac{5}{2}\times(4\cos^2 2t + 4\sin^2 2t + 4) \\
&= \frac{5}{2}\times(4+4) \\
&= 20\,\text{J}.
\end{aligned}
$$

(b) The work done is given by

$$
\begin{aligned}
\text{WD} &= \int \mathbf{F}.\mathbf{v}\,dt \\
&= \int 5(-4\sin 2t\,\mathbf{i} - 4\cos 2t\,\mathbf{j}).(2\cos 2t\,\mathbf{i} - 2\sin 2t\,\mathbf{j} + 2\,\mathbf{k})\,dt \\
&= \int 5(-8\sin 2t\cos 2t + 8\sin 2t\cos 2t + 0)\,dt \\
&= 0.
\end{aligned}
$$

Exercise 6G

1. The force vector **F** $(3\mathbf{i}+\mathbf{j}+7\mathbf{k})$ moves a particle from the origin to the position vector $(6\mathbf{i}+2\mathbf{j}-\mathbf{k})$. Find the work done by the force **F**.

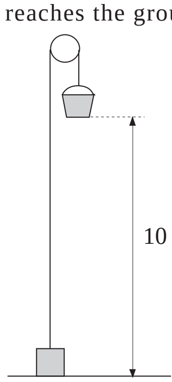
2. Three forces $\mathbf{A}=2\mathbf{i}+4\mathbf{j}-\mathbf{k}$, $\mathbf{B}=4\mathbf{i}-7\mathbf{j}-4\mathbf{k}$ and $\mathbf{C}=3\mathbf{i}+5\mathbf{j}-4\mathbf{k}$ are acting on a particle which is displaced from $3\mathbf{i}+2\mathbf{j}+\mathbf{k}$ to $15\mathbf{i}+6\mathbf{j}+3\mathbf{k}$.
   (a) Find the work done by each of the forces.
   (b) Find the resultant force and the work done by this force .

3. A mass m moves such that its position vector **r** is given by $(\frac{1}{3}t^3\mathbf{i}+\frac{1}{2}t^2\mathbf{j}+tk)$. Find
   (a) the kinetic energy of the mass;
   (b) the force acting on the mass;
   (c) the power exerted by this force.

4. A force $(-\mathbf{i}+3\mathbf{j})$ acts on a body of mass 5 kg. If the body has an initial velocity $(2\mathbf{i}-\mathbf{j})$ show, by considering the impulse by the force, that 5 s later the body is moving at right angles to its initial direction.

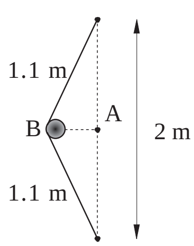
6.9 Miscellaneous Exercises

1. On a building site a bucket of mass m, attached to an inextensible rope over a pulley is 10 m above the ground. A counterweight whose mass is $\frac{3}{4}$ that of the bucket is attached to the other end of the rope at ground level. If the system is released from rest what is the speed of the bucket when it reaches the ground?



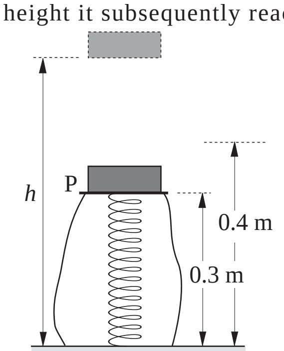
2. A mass m is suspended from a fixed point A by an elastic string of natural length l and stiffness 4 mg. The mass is pulled down a distance d from its equilibrium position and then released. If it just reaches A find d.

3. An archaeologist investigates the mechanics of large catapults used in sieges of castles. The diagram shows a simplified plan of such a catapult about to be fired horizontally.

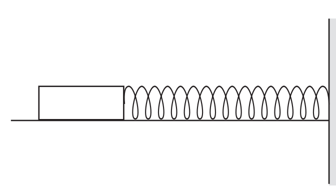


The rock B of mass 20 kg is in the catapult as shown. Calculate the speed with which the rock is released at A when the elastic string returns to its natural length of 2 m if the string's stiffness is 250 Nm^{-1}

4. A platform P has negligible mass and is tied down so that the 0.4 m long cords keep the spring compressed 0.6 m when nothing is on the platform. The stiffness of the spring is 200 Nm^{-1}. If a 2 kg block is placed on the platform and released when the platform is pushed down 0.1 m, determine the velocity with which the block leaves the platform and the maximum height it subsequently reaches.



5. A 10 kg block rests on a rough horizontal table. The spring, which is not attached to the block, has a stiffness $k=500\ \text{Nm}^{-1}$. If the spring is compressed 0.2 m and then released from rest determine the velocity of the block when it has moved through 0.4 m. The coefficient of friction between the block and the table is 0.2.



6. For a ball projected at speed u at an angle θ to the horizontal the speed v at a height y is given by $v^2 = u^2 - 2gy$.

   Use the conservation of energy to derive this result.

7. An elastic string AB, of natural length l and modulus mg is fixed at one end A and to the other is attached a mass m. The mass hangs in equilibrium vertically below A. Determine the extension of the string. The mass is now pulled vertically downwards a further distance $\frac{1}{2}l$ and released from rest. Show that if x is the extension of the string from the equilibrium position the energy equation is

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - mg(2l+x) + \frac{mg(l+x)^2}{2l} = \text{constant}.$$

   Differentiating the equation with respect to t show that

$$\frac{d^2x}{dt^2} = -\frac{g}{l}x$$

   and hence determine the acceleration of the mass when it is $\frac{7l}{4}$ below A.

   Find the maximum values of velocity and acceleration.

8. A wagon of mass 2.5×10^3 kg travelling at 2 ms^{-1} is brought to rest by a buffer (a spring) having a stiffness of $5\times10^5\,\text{Nm}^{-1}$. Assuming that the wagon comes into contact smoothly without rebound, calculate the deflection of the buffer if there is a constant friction force of 7500 N. Discuss the effect of friction.

9. (In this question you should assume g is 9.8 ms^{-2})

   The tension in a light elastic spring is given by Hooke's Law when the extension of the spring is x metres, where $0 \le x \le 0.5$. When $x = 0.5$ the tension is 7.35 N. Find the work done in increasing x from 0.2 to 0.5.

   For $x > 0.5$ the tension is not given by Hooke's Law. Values of the tension T Newtons, for specific values of x in the range $0.5 \le x \le 0.9$ are given in the following table.

x	0.5	0.6	0.7	0.8	0.9
T	7.35	9.03	10.80	12.90	15.03

   Using Simpson's rule, with five ordinates, show that the work done in increasing x from 0.5 to 0.9 is approximately 4.39 J.

   A particle P is attached to one end of the spring and is suspended with the other end of the spring attached to a fixed point. Given that the particle is in equilibrium when $x = 0.2$, show that the mass of the particle is 0.3 kg.

   When the particle is at rest in the equilibrium position an impulse is applied to it so that it moves vertically downwards with an initial speed of 6 ms^{-1}. Given that the work done in extending the spring from $x = 0.5$ to $x = 0.9$ is exactly 4.39 J find the square of the speed of the particle when x is first equal to 0.9.

   (AEB)

10. (In this question you should assume g is 9.8 ms^{-2})

   When a car is moving on any road with speed v ms^{-1} the resistance to its motion is $(a + bv^2)$ N, where a and b are positive constants. When the car moves on a level road, with the engine working at a steady rate of 53 kW, it moves at a steady speed of 40 ms^{-1}. When the engine is working at a steady rate of 24 kW the car can travel on a level road at a steady speed of 30 ms^{-1}. Find a and b and hence deduce that, when the car is moving with speed 34 ms^{-1}, the resistance to its motion is 992 N.

   Given that the car has mass 1200 kg find, in ms^{-2} correct to 2 decimal places, its acceleration on a level road at the instant when the engine is working at a rate of 51 kW and the car is moving with speed 34 ms^{-1} so that the resistance to the motion is 992 N.

   The car can ascend a hill at a steady speed of 34 ms^{-1} with the engine working at a steady rate of 68 kW. Find, in degrees correct to one decimal place, the inclination of the hill to the horizontal.

   (AEB)

11. A ball B of mass m is attached to one end of a light elastic string of natural length a and modulus $2mg$. The other end of the string is attached to a fixed point O. The ball is projected vertically upwards from O with speed $\sqrt{(8ga)}$;

   find the speed of the ball when $\text{OB} = 2a$.

   Given that the string breaks when $\text{OB} = 2a$, find the speed of the ball when it returns to O.

   (AEB)

12. (In this question you should assume g is 9.8 ms^{-2})

   A lorry has mass 6 tonnes (6000 kg) and its engine can develop a maximum power of 10 kW. When the speed of the lorry is v ms^{-1} the total non-gravitational resistance to motion has magnitude $25v$ N. Find the maximum speed of the lorry when travelling

   (a) along a straight horizontal road,

   (b) up a hill which is inclined at an angle $\sin^{-1}(1/100)$ to the horizontal, giving your answer to 2 decimal places.

   (AEB)

13. The position vector **r**, relative to a fixed origin O, at time t of a particle P is given by

$$\mathbf{r} = a(\omega t - \sin \omega t)\mathbf{i} + a(\omega t + \cos \omega t)\mathbf{j},$$

where a and ω are positive constants. Determine the velocity and acceleration of P at time t and show that the acceleration is of constant magnitude.

Given that the particle is of mass m, find

(a) the force acting on the particle,

(b) the value of t in the interval $0 \le t \le \dfrac{\pi}{\omega}$ when the force is perpendicular to the velocity,

(c) the kinetic energy when $t = \dfrac{\pi}{4\omega}$ and when $t = \dfrac{\pi}{\omega}$,

(d) the work done by the force acting on the particle in the interval $\dfrac{\pi}{4\omega} \le t \le \dfrac{\pi}{\omega}$.

(AEB)

14. Prove, by integration, that the work done in stretching an elastic string, of natural length l and modulus of elasticity λ, from length l to a length $l + x$ is $\dfrac{\lambda x^2}{2l}$.

A particle of mass m is suspended from a fixed point O by a light elastic string of natural length l. When the mass hangs freely at rest the length of the string is $\dfrac{13l}{12}$. The particle is now held at rest at O and released. Find the greatest extension of the string in the subsequent motion.

By considering the energy of the system when the length of the string is $l + x$ and the velocity of the particle is v explain why

$$\frac{1}{2}mv^2 = mg(l + x) - 6mg\frac{x^2}{l}.$$

Hence show that the kinetic energy of the particle in this position may be written as

$$\frac{mg}{l}\{\alpha l^2 - 6(x - \beta l)^2\},$$

where α and β are positive constants which must be found. Hence deduce that the maximum kinetic energy of the particle during the whole of its motion occurs when it passes through the equilibrium position.

(AEB)

15. One end of an elastic string of modulus 20 mg and natural length a is attached to a point A on the surface of a smooth plane inclined at an angle of $30°$ to the horizontal. The other end is attached to a particle P of mass m. Initially P is held at rest at A and then released so that it slides down a line of greatest slope of the plane. By use of conservation of energy, or otherwise, show that the speed v of P when $AP = x, (x > a)$, is given by

$$v^2 = \frac{g}{a}(41ax - 20a^2 - 20x^2).$$

(a) Find the maximum value of v in the subsequent motion.

(b) Find the maximum value of x in the subsequent motion. (AEB)

16. A train of mass 10^6 kg starts from rest and moves up a slope which is inclined at an angle arcsin (1/250) to the horizontal. Throughout its motion, the train's engines work at a power of 10^6 W and the train experiences a variable resistance. After 15 minutes, the train is moving with speed 20 ms^{-1} and has travelled a distance of 15 km along the slope. Taking $g = 9.8\ ms^{-2}$, find the increase in the kinetic and potential energies of the train during this 15 minute period and hence deduce the total work done against the resistance during this period. (AEB)

17. Two particles A and B, of mass $3m$ and $2m$ respectively, are attached to the ends of a light inextensible string which passes over a smooth fixed peg. The particles move freely in a vertical plane with both the hanging parts of the string vertical. Find the magnitude of the common acceleration of the particles and the tension in the string.

When the particle A is moving downwards with speed v it hits a horizontal inelastic table so that its speed is immediately reduced to zero. Assume that B never hits the peg. Determine, in terms of v, m and g as appropriate,

(a) the time that A is resting on the table after the first collision and before it is jerked off,

(b) the speed with which A is first jerked off the table,

(c) the difference between the total kinetic energy immediately before A first hits the table and the total kinetic energy immediately after A starts moving upwards for the first time,

(d) the difference between the total kinetic energy immediately before A first hits the table and the total kinetic energy just after A starts moving upwards for the second time.

(AEB)

7 CIRCULAR MOTION

Objectives

After studying this chapter you should

- appreciate that circular motion requires a force to sustain it;
- know and understand that, for motion in a circle with uniform angular velocity, the acceleration and the force causing it are directed towards the centre of the circle;
- use your knowledge to model applications of circular motion.

7.0 Introduction

You will have seen and experienced examples of circular motion throughout your everyday life. Theme park rides such as ‘The Wave Swinger’, ‘The Pirate Ship’, ‘Loop-the-Loop'; cornering on a bicycle or in a car; household equipment such as washing machines, tumble and spin driers, salad driers; specialised devices such as the centrifuge in the chemistry laboratory; the governors of a steam engine. These are just a few of the many examples which you will have come across or been a part of.

The purpose of this chapter is to enable you to analyse circular motion, to help you model some of the situations noted above and be able to predict and explain observations which you can make.

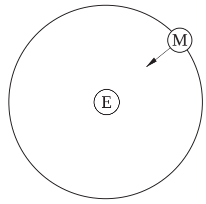
The philosophers of the ancient world considered circular motion to be a natural motion. The heavenly bodies, the planets and the stars, moved in circles around the Earth. Once set upon their paths by the Gods, these bodies continued to move in circles without any further intervention. No force was required to sustain their heavenly orbits.

However, from Newton’s point of view, circular motion requires a force to sustain it. Recall Newton’s First Law from Chapter 2,

> A body remains in a state of rest or moves with uniform motion, unless acted upon by a force.

A body describing a circle is certainly not at rest. Nor does it move with constant velocity, since, whether the speed is changing or constant, the direction of the motion is changing all the time. From Newton’s First Law, there must be a resultant force acting on the body and Newton’s Second Law equates this force to the mass times acceleration, so the body must be accelerating.

An example of this is the case of the Moon (M in diagram opposite) orbiting the Earth (E). The Moon describes a path around the Earth which is approximately circular. The force which the Earth exerts upon the Moon is the force of gravity and this pulls the Moon towards the Earth. If this force of gravity did not act, the Moon would move off at a tangent to its path around the Earth. Indeed Newton envisaged the motion of the Moon around the Earth as a series of steps, a tangential movement followed by a move in towards the Earth. If you make the time intervals sufficiently small, this saw-tooth curve becomes a circle.



A further consequence is that since the force upon the Moon is directed towards the Earth so the acceleration is also in this direction.

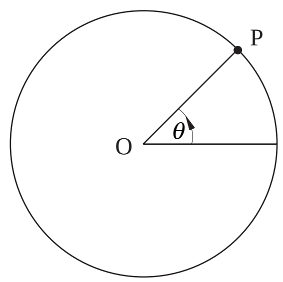
A body moving in a circle must

(a) from Newton's First Law, have a resultant force acting on it;

(b) from Newton's Second Law, be accelerating in a specific direction.

7.1 Angular velocity and angular speed

As a particle P moves around the circle, centre O, the radius OP turns through an angle θ, measured from a fixed radius. The conventions are



(a) anti-clockwise rotation is positive,

(b) θ is measured in radians.

The rate at which θ is changing with respect to time is $\frac{d\theta}{dt}$.

This is called the **angular velocity** of the particle P.

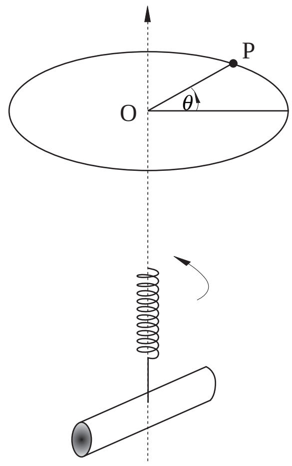
The angular speed is ω, the magnitude of $\frac{d\theta}{dt}$, and is measured in radians per second.

$$\omega = \left|\frac{d\theta}{dt}\right|.$$

You may find it strange that $\frac{d\theta}{dt}$ is called the angular velocity.

The word velocity conjures up thoughts of a vector and yet $\frac{d\theta}{dt}$ is not written as a vector, nor does it seem to have a direction associated with it.

However, think of a corkscrew below the plane of the circle and pointing vertically upwards towards the centre of the circle. As the corkscrew rotates anti-clockwise, i.e. right-handedly, when viewed from above, it will advance along the vertical axis towards and through the centre of the circle. It is this direction of motion of the corkscrew that is the direction associated with the angular velocity $\frac{d\theta}{dt}$.

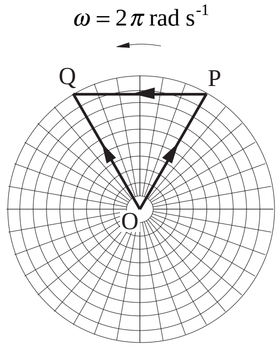


Activity 1 The relation between velocity and angular velocity

For this activity you will need a piece of polar graph paper, a ruler, a protractor, or a piece of polar graph paper and a programmable calculator.

A piece of polar graph paper can be used to represent the path of the particle moving in a circle with constant angular speed, the radius of the circle sweeping out equal angles in equal intervals of time.

If the angular speed of the particle is 2π rad s^{-1}, then it completes one revolution every second and each of the points at the ends of the 36 radii drawn represent the positions of the particle every $\frac{1}{36}$ of a second. (Remember: 2π radians $\equiv 360°$.)



In the diagram opposite, the angle POQ is $\frac{\pi}{3}$ radians and the particle has taken $\frac{6}{36} = \frac{1}{6}$ seconds to travel from P to Q around the circle.

The average velocity of the particle is

$$\frac{\overrightarrow{OQ} - \overrightarrow{OP}}{t} = \frac{\overrightarrow{PQ}}{t}$$

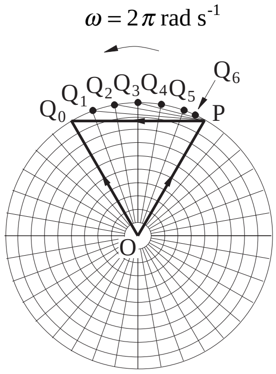
where $t = \frac{1}{6}$ seconds, the time taken to travel from P to Q.

The magnitude of the average velocity of the particle is

$$\left|\frac{\overrightarrow{PQ}}{t}\right| = \frac{PQ}{t}.$$

$\frac{PQ}{t}$ **is not the average speed of the particle. Why not?**

Now investigate the average velocity of the particle for different times t.



Copy and complete the table below, using the diagram opposite.

angle swept out	time in secs	distance PQ	$\frac{PQ}{t}$ ms^{-1}
$\frac{\pi}{3}$	$\frac{1}{6}$	$PQ_0 =$	
$\frac{2\pi}{18}$	$\frac{5}{36}$	$PQ_1 =$	
$\frac{2\pi}{9}$			
$\frac{\pi}{6}$			
$\frac{\pi}{9}$			
$\frac{\pi}{18}$			
$\frac{\pi}{36}$			

What value does $\frac{PQ}{t}$ **approach as** t **gets smaller?**

The average velocity, $\frac{\overrightarrow{PQ}}{t}$, as t becomes smaller and smaller approaches the velocity of the particle at P.

What is the magnitude of the velocity at P?

Is there a relation between this magnitude, ω**, and** r**, the radius of the circle in metres?**

In which direction is the velocity at P?

Exercise 7A

1. What is the angular speed of the tip of the minute hand of a clock in:
   (a) revolutions per minute;
   (b) degrees per second;
   (c) radians per second?
2. A particle has a constant angular speed ω and is moving in a circle of radius r.
   (a) What is the time for one revolution?
   (b) How far has the particle travelled in one revolution?
   (c) What is the average speed of the particle?
3. Estimate the angular speed of the Moon as it orbits the Earth.

   The distance of the Moon from the Earth is approximately 355,000 km. Calculate the speed of the Moon relative to the Earth.
4. Estimate the angular speed of the Earth as it orbits the Sun. (Assume that the orbit is a circular path.)

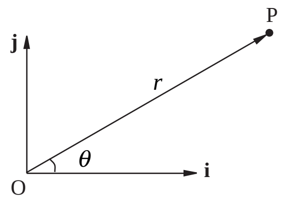
   The distance of the Earth from the Sun is approximately 150×10^6 km. Calculate the speed of the Earth relative to the Sun.

7.2 Describing motion in a circle with vectors

Activity 1 suggests that for a particle moving in a circle with constant angular speed ω, the velocity of the particle at any instant is of magnitude $r\omega$ and is directed along the tangent to the circle at the position of the particle. This section begins with a proof of this result.

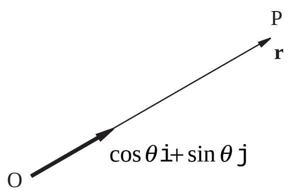
A particle P, on a circle of radius r, has coordinates:

$$(r\cos\theta,\ \ r\sin\theta) \qquad (1)$$



where O is the origin and the centre of the circle; θ is the angle, measured in radians, which the radius OP makes with the positive direction of the x-axis.

As P moves around the circle r remains constant and θ varies with t, the time. At any instant of time the position vector of P with respect to the origin O referred to the axes shown is,



$$\mathbf{r} = \mathbf{OP} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$$

$$\Rightarrow \qquad \mathbf{r} = r(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \qquad (2)$$

The magnitude of the vector $\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is $\sqrt{\cos^2\theta + \sin^2\theta} = 1$, so this vector is the unit vector in the direction of the radius OP and rotates with P as P moves round the circle. The position vector

$$\mathbf{r} = r(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$$

then has magnitude r and is in the direction of the unit vector

$(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$.

The velocity, $\mathbf{v}$, of the particle is then given by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\big(r(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\big)$$

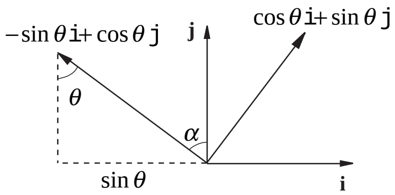
$$= r\left(\frac{d}{dt}(\cos\theta)\,\mathbf{i} + \frac{d}{dt}(\sin\theta)\,\mathbf{j}\right)$$

$$= r\left(\frac{d}{d\theta}(\cos\theta)\frac{d\theta}{dt}\,\mathbf{i} + \frac{d}{d\theta}(\sin\theta)\frac{d\theta}{dt}\,\mathbf{j}\right)$$

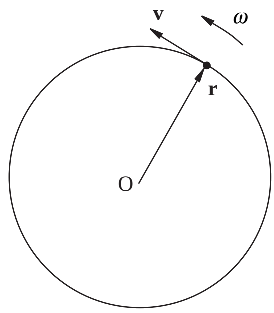
using the 'function of a function' rule for differentiation.

Hence, $$\mathbf{v} = r\left(-\sin\theta\frac{d\theta}{dt}\,\mathbf{i} + \cos\theta\frac{d\theta}{dt}\,\mathbf{j}\right) \qquad (3)$$

Again, $(-\sin\theta)^2 + (\cos\theta)^2 = 1$, thus the vector $(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})$ is also a unit vector, but what is its direction?



From the diagram opposite, it is clear that $\alpha = \theta$ so that the vector $-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$ is perpendicular to the vector $\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ This means that the direction of the velocity **v** is perpendicular to the direction of the radius vector **r**.



Hence the velocity at P is along the tangent to the circle in the direction θ increasing when $\frac{d\theta}{dt}$ is positive and in the opposite direction when $\frac{d\theta}{dt}$ is negative.

The magnitude of the velocity **v**, is

$$|\mathbf{v}| = v = \left| r\frac{d\theta}{dt}(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) \right|$$

$$= r\left|\frac{d\theta}{dt}\right|$$

$$= r\omega.$$

The results you should have obtained from Activity 1 are therefore validated.

Summary

The velocity of a particle P moving in a circle of radius r with angular speed ω is of magnitude $r\omega$ and is along the tangent to the circle at P in the direction of θ increasing when $\frac{d\theta}{dt}$ is positive and in the opposite direction when $\frac{d\theta}{dt}$ is negative.

In the introduction to this chapter you saw that any particle moving in a circle must be accelerating. Using the system of unit vectors which has just been set up, this acceleration can now be found.

From equation (3) above,

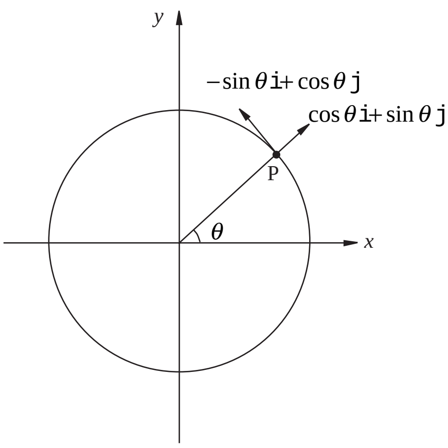
$$\mathbf{v} = r\frac{d\theta}{dt}\left(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\right).$$

Hence the acceleration, **a**, is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left\{r\frac{d\theta}{dt}\left(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\right)\right\}$$

$$= r\frac{d\theta}{dt}\frac{d}{dt}\left(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\right) + \left(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\right)\frac{d}{dt}\left(r\frac{d\theta}{dt}\right)$$

using the product rule for differentiation. (Remember that r is constant.).



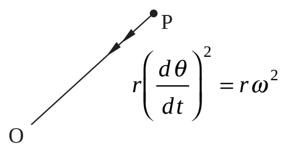
Hence

$$\mathbf{a} = r\frac{d\theta}{dt}\left(-\cos\theta\frac{d\theta}{dt}\mathbf{i} - \sin\theta\frac{d\theta}{dt}\mathbf{j}\right) + r\frac{d^2\theta}{dt^2}\left(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\right)$$

using the function of a function rule for differentiation, and so

$$\mathbf{a} = -r\left(\frac{d\theta}{dt}\right)^2\left(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}\right) + r\frac{d^2\theta}{dt^2}\left(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\right).$$

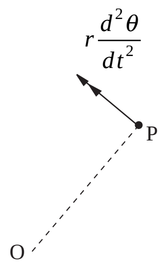
This expression is the sum of two vectors. One of these, $-r\left(\frac{d\theta}{dt}\right)^2\left(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}\right)$, is associated with the unit vector $\left(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}\right)$ which is in the direction from O to P.



Since $\left(\frac{d\theta}{dt}\right)^2$ is positive, $-r\left(\frac{d\theta}{dt}\right)^2$ is negative and hence the vector is in the direction from P to O, that is radially inwards.

Its magnitude is $r\left(\frac{d\theta}{dt}\right)^2$, which is equal to $r\omega^2$.

Since $v = r\omega$, this can be written as $\frac{v^2}{r}$.



The second vector, $r\frac{d^2\theta}{dt^2}(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})$, is associated with the unit vector $(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})$, which is parallel to the tangent at P. Its magnitude is $r\left|\frac{d^2\theta}{dt^2}\right|$.

In the case when $\frac{d\theta}{dt}$, and hence the angular speed $\omega = \left|\frac{d\theta}{dt}\right|$ are constant, $\frac{d^2\theta}{dt^2} = 0.$

Summary

When a particle is moving in a circle with constant angular speed ω, there is only one component of acceleration. This is $r\omega^2$, and it is directed towards the centre of the circle.

Example

A car has wheels which are 75 cm in diameter and is travelling at a constant speed of 60 kph. If there is no slipping between the wheels and the road, calculate the angular speed and the acceleration of a stone wedged in the tread of one of the tyres.

Solution

Converting the speed to ms^{-1} gives

$$v = 60 \text{ kph } = \frac{60 \times 1000}{60 \times 60} \text{ ms}^{-1}$$

$$= 16.\dot{6} \text{ ms}^{-1} \text{ (to 3 sig. fig.)}$$

The stone describes circular motion with constant angular speed, so $v = r\omega$, and since $r = 37.5 \text{ cm} = 0.375 \text{ m}$, then

$$\omega = \frac{v}{r} = 44.3 \text{ rads s}^{-1} \quad (\text{to 3 sig. fig.}).$$

For circular motion with constant angular speed ω, the acceleration is $r\omega^2$ radially inwards; hence

$$a = 736 \text{ ms}^{-2}.$$

How might your answer be changed if there is slipping between the wheels and the road?

Example

A particle is moving in a circle of radius 2 m such that the angle swept out by the radius is given by $\theta = t^2$. Calculate the angular velocity, the speed and the acceleration of the particle, giving your answers in terms of t.

Solution

Since $\theta = t^2$,

differentiating θ gives the angular speed of the particle, ω, as

$$\omega = \frac{d\theta}{dt} = 2t \text{ rads s}^{-1}.$$

The actual speed of the particle, v, is given by

$$v = r\omega.$$

Substituting for $r = 2$ and $\omega = 2t$ gives

$$v = 4t \text{ ms}^{-1}.$$

The acceleration has two components since the angular speed is not constant; these components are

$$\text{radial component} = r\omega^2 = 8t^2,$$

$$\text{tangential component} = \frac{rd^2\theta}{dt^2} =, \text{ using } \frac{d^2\theta}{dt^2} = 2.$$

Hence the acceleration is $8t^2 \text{ ms}^{-2}$ directed radially inwards and 4 ms^{-2} tangentially in the direction of increasing θ.

xercise 7B

1. A bicycle wheel has a diameter of 90 cm and is made to turn so that a spoke sweeps out an angle θ in time t given by $\theta = 2t$ radians. The bicycle is stationary. Calculate the angular velocity, the speed and the acceleration of a point on the rim of the wheel.
2. The acceleration of the valve of a tyre, which is rotating, is known to be radially inwards and of magnitude 120 ms^{-2}.
   If the distance of the valve from the centre of the hub of the wheel is 40 cm, calculate the angular speed of the wheel and the speed of the valve.
3. In each of the following cases, the angle θ swept out in time t for a particle moving in a circle of radius 1 m is given as a function of t. Calculate the components of velocity and acceleration in each case:

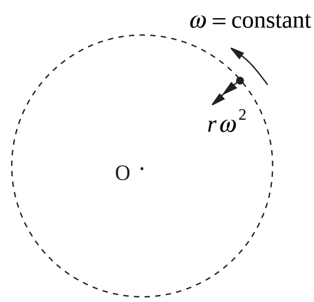
   (a) $\theta = t$ (b) $\theta = t^3$.
4. A record turntable has a radius of 15 cm and takes 4 seconds to reach an angular speed of 33 rpm from rest. Assuming that the angular speed, ω, increases at a constant rate, express ω as a function of t and calculate the components of acceleration of a point on the rim of the turntable after 2 seconds.

7.3 Motion in a circle with constant angular speed

In section 7.2 you saw that the acceleration of a particle P moving in a circle of radius r metres with constant angular speed

$$\omega \text{ rad s}^{-1} \text{ has magnitude } r\omega^2 \text{ or } \frac{v^2}{r} \text{ and is directed}$$

inwards along the radius from P towards the centre of the circle.



From Newton's Second Law you can conclude that the resultant force on the particle must also act in this direction and has magnitude $mr\omega^2$ or $\frac{mv^2}{r}$, where m is the mass of the particle.

Activity 2 Investigating the forces in circular motion

For this activity you will need a circular cake tin or a plastic bucket and a marble.

Place the marble inside the cake tin.

Holding the cake tin horizontally, move it around so that the marble rolls around the sides of the tin, in contact with the base.

You should be able to get the marble going fast enough so that for a short period of time you can hold the tin still and the marble will describe horizontal circles with uniform angular velocity.

What do you feel through your hands?

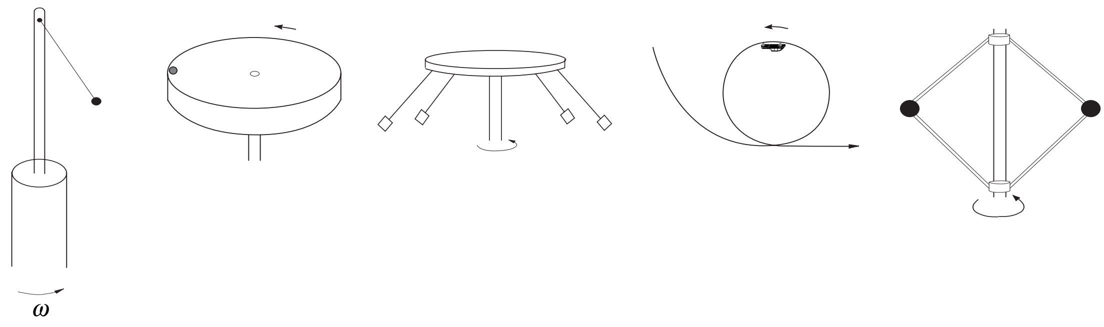
What are the forces acting on the marble?

Draw a force diagram for the forces acting on the marble.

What is the magnitude and direction of the resultant force acting on the marble?

Activity 3 Force and acceleration

For each of the following situations, mark on the diagram the forces which are acting on the particle, the direction of the resultant force and the acceleration.



Example

The marble of Activity 2 has mass 10 g and the cake tin a radius of 15 cm. If the marble rolls round the bottom of the tin in contact with the side at an angular speed of 20 rad s^{-1}, what are the normal contact forces of the side and base of the tin on the marble?

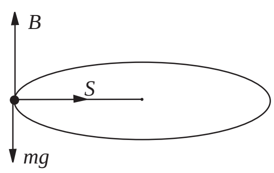
Solution

The forces on the marble are:

its weight $0.01\, g$ N vertically downwards;

the normal contact force B newtons of the base on it;

the normal contact force S newtons of the side on it.



The acceleration of the marble is radially inwards and has magnitude

$$\omega^2 r = (20)^2 \times 0.15 = 60 \text{ ms}^{-2}.$$

Applying Newton's Second Law in the radial direction gives

$$S = m\omega^2 r = 0.01 \times 60 = 0.6 \text{ N}.$$

Since there is no acceleration vertically,

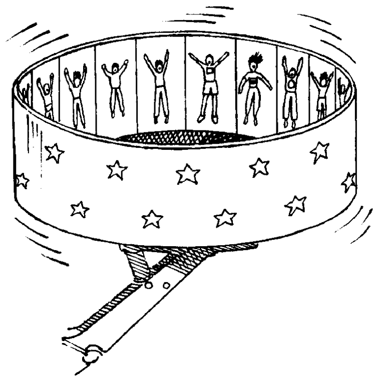
$$B - 0.01\,g = 0,$$

so $$B = 0.1\text{ N}.$$

The rotor ride

What are the forces acting on the people in the ride shown opposite?

Is there a critical angular speed at which this ride must operate to achieve the effects on the people which the diagram shows?



Modelling the ride

Set up the model by considering each person as a particle, and looking at the forces acting upon it. You looked at a physical situation very similar to this in Activity 2, so you should have some idea of the forces acting.

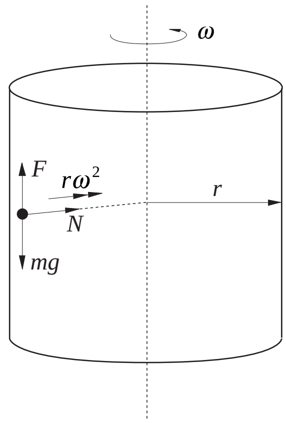
The forces acting are:-

(a) the weight, mg, acting vertically downwards;

(b) the force of friction, F, acting vertically upwards and preventing the particle sliding down the wall;

(c) the normal contact force, N, exerted radially inwards by the wall on the particle.

The particle is moving in a circle, radius r, with constant angular speed ω, so there is an acceleration $r\omega^2$ directed along the radius towards the centre of the circle.



Considering the forces in the two mutually perpendicular directions, vertically and radially, and applying Newton's Second Law in each direction you have,

vertically $F - mg = 0,$

radially $N = mr\omega^2.$

But for equilibrium, $F \le \mu N$

and therefore $mg \le \mu m r\omega^2$

$$\Rightarrow \quad \omega^2 \ge \frac{g}{r\mu}.$$

Therefore there is a critical angular speed,

$$\omega = \sqrt{\frac{g}{r\mu}},$$

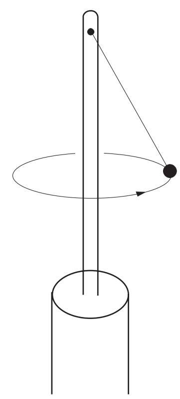
which is independent of the mass of the particle or person. This is clearly shown by the people on the ride itself. No matter what their size is, they all behave in exactly the same way. The angular speed, ω, must be greater than the critical angular speed above, for the effects to be observed.

For a particular ride, the diameter of the drum is about 10 metres and the coefficient of friction is 0.5. If g is taken to be $10\ \text{ms}^{-2}$, then the critical angular speed

$$\omega = \sqrt{\frac{10}{(5\times 0.5)}} = 2 \text{ rad s}^{-1}$$
$$= 19 \text{ rpm.}$$

Therefore the ride will have to be rotating in excess of this angular speed for the effect of people 'sticking' to the wall to be observed.

Activity 4 The conical pendulum

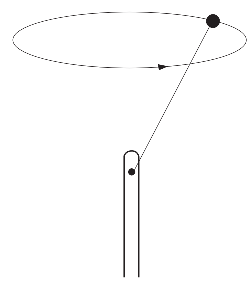


For this activity you will need either a mass on the end of a piece of string or a mechanics kit containing a conical pendulum.

A bob is made to describe circles which are in a horizontal plane. If you have access to a suitable mechanics kit you can set it up as shown opposite. If not, then you can do much the same thing using a mass on the end of a piece of string. This particular form of motion is often referred to as the 'conical pendulum'.

If the angular speed increases, what happens to the bob?

Does the mass of the bob have any effect?



Is it possible to get the bob to describe:

(a) a circle so that the string is horizontal?

(b) a circle above the point of suspension rather than below?

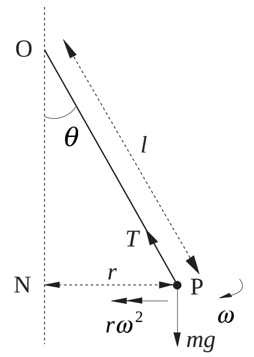
To model this situation, what assumptions would you make?

Modelling the conical pendulum

Set up the model

To begin, the following assumptions simplify the problem;

1. the bob is a particle;
2. the string is light and inextensible;
3. the point of suspension is directly above the centre of the horizontal circle described by the bob;
4. the angular speed is constant.



With these assumptions the problem becomes one of motion of a particle in a circle of radius r about the axis of rotation ON, with constant angular speed ω. The length of the string is l and it makes an angle θ to the axis of rotation.

The forces acting on the particle are the tension T and the weight mg.

Since the particle describes circular motion, it has an acceleration $r\omega^2$ towards the centre of the circle N.

(P describes a horizontal circle with centre at N.)

Applying Newton's Second Law vertically and horizontally gives two equations of motion:

$$T\cos\theta - mg = 0 \qquad (1)$$

$$T\sin\theta = mr\omega^2. \qquad (2)$$

Solving the equations

The equations are solved to give an expression for θ.

From triangle ONP, the radius r is given by

$$r = l\sin\theta.$$

Substituting for r into equation (2) gives

$$T\sin\theta = ml\omega^2\sin\theta$$

hence $$T = ml\omega^2.$$

Substituting for T into equation (1) gives

$$ml\omega^2 \cos\theta = mg$$

hence $$\cos\theta = \frac{g}{l\omega^2}. \qquad (3)$$

Validating the solution

The questions raised in Activity 4 , relating to the conical pendulum, can now be explored mathematically using the solution in equation (3).

Activity 5 The conical pendulum revisited

What happens to $\cos\theta$ as ω changes?

Does the mass affect the angle θ?

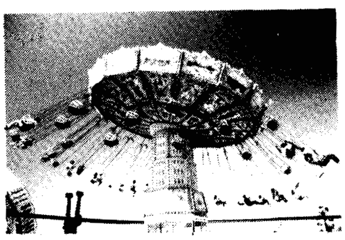
What is the maximum value of θ? Is it realistically possible to achieve this value?

Find the minimum value of the angular speed ω for the mass to describe circular motion.

Do your answers to Activity 3 agree with the mathematical model?

Activity 6 The chair-o-planes ride

Look carefully at the picture of the chair-o-planes ride.



What might happen if the solution depended on mass?

How would you amend the model of the conical pendulum to model this ride?

Find a formula for the angle that each supporting rope makes with the vertical.

It may be possible to visit a fair or a park and, by calculating some distances, estimate the angular speed of the ride.

Exercise 7C

1. A particle of mass 2 kg is moving in a circle of radius 5 m with a constant speed of $3\ \text{ms}^{-1}$. What is the magnitude and direction of the resultant force acting on the particle?
2. A penny is placed on the turntable of a record player $0.1\ \text{m}$ from the centre. The turntable rotates at 45 rpm. If the penny is on the point of slipping, calculate the coefficient of friction between the penny and the turntable.

   Calculate the resultant force acting on the penny in terms of *m*, the mass of the penny, when the turntable is rotating at 33 rpm.
3. An inextensible string has a length of 3 m and is fixed at one end to a point O on a smooth horizontal table. A particle of mass 2 kg is attached to the other end and describes circles on the table with O as centre and the string taut. If the string breaks when the tension is 90 N, what is the maximum speed of the particle?
4. A particle of mass 4 kg is attached by a light inextensible string of length 3 m to a fixed point. The particle moves in a horizontal circle with an angular speed of $2\ \text{rads}^{-1}$.

   Calculate:

   (a) the tension in the string;

   (b) the angle the string makes with the vertical;

   (c) the radius of the circle.
5. A particle of mass 5 kg is attached to a fixed point by a string of length 1 m. It describes horizontal circles of radius 0.5 m. Calculate the tension in the string and the speed of the particle.
6. A bead of mass *m* is threaded on a string of length 8 m which is light and inextensible. The free ends of the string are attached to two fixed points separated vertically by a distance which is half the length of the string, the lower fixed point being on the horizontal table. The bead is made to describe horizontal circles on the table around the lower fixed point, with the string taut. What is the maximum value of ω, the angular speed of the bead, if it is to remain in contact with the table?
7. A geostationary satellite orbits the Earth in such a way that it appears to remain stationary over a fixed point on the equator. The satellite is actually orbiting the Earth.

   How long does one orbit take?

   What is the radius of the orbit?

   How far away from the surface of the Earth is the satellite?

   What is the speed of the satellite?

   Does the mass of the satellite matter?

   (The Universal Constant of Gravitation,
   $G = 6.67 \times 10^{-11}\ \text{Nm}^2\ \text{kg}^{-2}$.
   Mass of the Earth $= 6 \times 10^{24}$ kg.
   Radius of the Earth $= 6 \times 10^{3}$ km.)

7.4 Motion on a banked curve

Why do birds need to bank in order to change the direction of their flight?

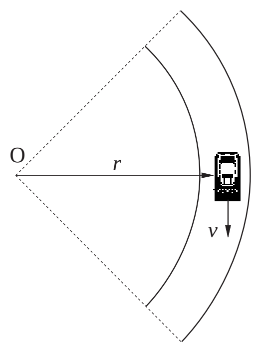
What advantages are gained?

Activity 7 Going round the bend

The diagram shows a car taking a bend at a constant speed *v*.

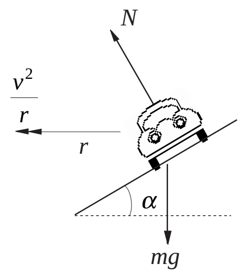
What is the greatest speed at which it can take the bend on a horizontal road?

Do you think that the car can take the bend at a greater speed if the bend is banked, than if it is not?



Modelling the motion of a car on a banked road

Assume that the car is a particle, the bend is a part of a circle and the road is inclined at an angle α to the horizontal.



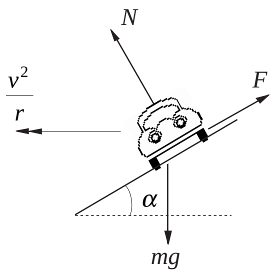
The forces acting on the car are its weight mg, vertically downwards, the normal contact force, perpendicular to the road surface, and the friction force F. However there is a need to stop and think for a moment about the friction force, F.

In which direction does it act?

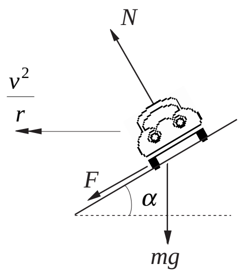
In which directions can it act?

Is it possible for the friction force to be zero?

If the car were on the point of sliding **down** the incline of the bend, F would act in the direction **up** the incline.



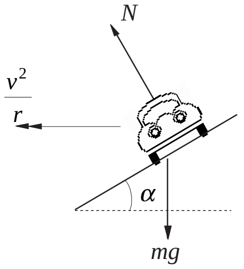
If the car were on the point of sliding **up** the incline, F would act in the direction **down** the incline.



Between these two extremes, F can be zero. This is important because of the reduced wear on the car tyres and the greater comfort for passengers and driver.

Consider the critical speed when friction is zero.

Taking the components of the forces acting on the car in two, mutually perpendicular directions, vertically and radially, and applying Newton's Second Law in each case,



vertically upwards, $$N\cos\alpha - mg = 0 \qquad (1)$$

radially inwards, $$N\sin\alpha = \frac{mv^2}{r}. \qquad (2)$$

Substituting for N in equation (2) from equation (1) gives

$$\frac{mg\sin\alpha}{\cos\alpha} = \frac{mv^2}{r}$$

$$\Rightarrow \qquad \tan\alpha = \frac{v^2}{rg} \qquad (3)$$

For a bend of given radius, this equation defines the correct angle of banking so that there is no friction force for a given speed. Alternatively, for a given angle of banking it defines the speed at which the bend should be taken.

Activity 8 Including the effects of friction

Suppose that the coeffficient of sliding friction between the car and the road is μ.

Find an expression for the constant speed of a car on a circular banked track of angle α to the horizontal:

(a) if the car is on the point of sliding **up** the incline;

(b) if the car is on the point of sliding **down** the incline.

When a railway engine takes the bend on a railway track, what is the effect on the rails of the flanges on the wheels?

Exercise 7D

1. A vehicle is approaching a bend which is of radius 50 m. The coefficient of friction between the road and the tyres is 0.5. Find the greatest speed at which the vehicle can safely negotiate the bend if it is horizontal.

   At what angle must the bend be banked if the vehicle is to negotiate the bend without any tendency to slip on the road at a speed of 80 kph?

2. A car is negotiating a bend of radius 100 m banked at an angle of $\tan^{-1}\left(\frac{3}{4}\right)$. What are the maximum and minimum speeds at which it can do this if the coefficient of friction between the road and the tyres is 0.5?

3. A light aeroplane is describing a horizontal circle of radius 200 m at a speed of 150 kph. Calculate the angle at which the plane is banked as it circles, stating the assumptions that you make.

4. One lap of a circular cycle track is 400 m, and the track is banked at $45°$. At what speed can the track be negotiated without any tendency for skidding or slipping to take place?

7.5 Miscellaneous Exercises

1. Lucy and Tom ride on a fairground roundabout. Lucy is 2 m and Tom is 1.5 m from the centre of rotation and the roundabout is rotating at 10 revolutions per minute. Find

   (a) the angular speed of the roundabout in rad s^{-1},

   (b) the speeds of Lucy and Tom.

2. A marble is made to rotate against the outside rim of a circular tray of radius 0.2 m. If the mass of the marble is 100 g and it is moving at 2 ms^{-1}, calculate the horizontal force that the marble exerts on the rim of the tray.

3. An athlete throwing the hammer swings the hammer in a horizontal circle of radius 2.0 m. If the hammer is rotating at 1 revolution per second, what is the tension in the wire attached to the hammer, if the mass of the wire is negligible and the mass of the hammer is 7.3 kg?

4. A car travels along a horizontal road and can travel without slipping at 40 kilometres per hour around a curve of radius 115 m. Find the coefficient of friction between the tyres and the road surface.

5. A train of mass 40 tonnes travels around a banked railway track which forms a circular arc of radius is 1250 m. The distance between the rail centres is 1.5 m and the outer rail is 37.5 mm vertically higher than the inner rail. If the train is travelling at 63 kilometres per hour, find the force on the side of the rail. (Use $g = 9.8\ \text{ms}^{-2}$).

6. An elastic string, of natural length l and modulus mg, has a particle of mass m attached to one end, the other end being attached to a fixed point O on a smooth horizontal table. The particle moves on the table with constant speed in a circle with centre O and radius $\frac{3}{2}l$. Find, in terms of g and l, the angular speed of the particle. (AEB)

7. A particle moves with constant angular speed around a circle of radius a and centre O. The only force acting on the particle is directed towards O and is of magnitude $\frac{k}{a^2}$ per unit mass, where k is a constant. Find, in terms of k and a, the time taken for the particle to make one complete revolution. (AEB)

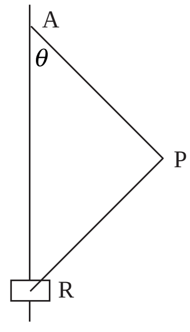
8. One end of light inextensible string of length l is fixed at a point A and a particle P of mass m is attached to the other end. The particle moves in a horizontal circle with constant angular speed ω. Given that the centre of the circle is vertically below A and that the string remains taut with AP inclined at an angle α to the downward vertical, find $\cos\alpha$ in terms of l, g and ω. (AEB)

9. A particle P is attached to one end of a light inextensible string of length 0.125 m, the other end of the string being attached to a fixed point O. The particle describes with constant speed, and with the string taut, a horizontal circle whose centre is vertically below O. Given that the particle describes exactly two complete revolutions per second find, in terms of g and π, the cosine of the angle between OP and the vertical. (AEB)

10. A particle P of mass m moves in a horizontal circle, with uniform speed v, on the smooth inner surface of a fixed thin, hollow hemisphere of base radius a. The plane of motion of P is a distance $\frac{a}{4}$ below the horizontal plane containing the rim of the hemisphere. Find, in terms of m, g and a, as appropriate, the speed v and the reaction of the surface on P.

    A light inextensible string is now attached to P. The string passes through a small smooth hole at the lowest point of the hemisphere, and has a particle of mass m hanging at rest at the other end. Given that P now moves in a horizontal circle on the inner surface of the hemisphere with uniform speed u and that the plane of the motion is now distant $\frac{a}{2}$ below the horizontal plane of the rim, prove that $u^2 = 3ga$. (AEB)

11. The figure shows a particle P, of mass m, attached to the ends of two light inextensible strings, each of length l. The other end of one string is attached to a fixed point A and the other end of the second string is attached to a ring R of mass $3m$. The ring can slide freely on a smooth vertical wire passing through A. The particle P describes a horizontal circle, whose centre is on the wire, with constant angular speed ω. The tensions in AP and PR are denoted by T_1 and T_2, respectively and the angle between AP and the vertical is denoted by θ. Given that the ring is at rest find T_2 in terms of m, g and θ and hence calculate the ratio T_1 / T_2.



Show that

$$T_1 + T_2 = m\omega^2 l.$$

Hence express $\cos\theta$ in terms of g, ω and l and find the range of possible values of ω. (AEB)

12. (In this question take g to be 9.8 ms^{-2})

A light elastic string of natural length 0.2 m has one end attached to a fixed point O and a particle of mass 5 kg is attached to the other end.

When the particle hangs at rest, vertically below O, the string has length 0.225 m. Find the modulus of elasticity of the string.

The particle is made to describe a horizontal circle whose centre is vertically below O. The string remains taut throughout this motion and is inclined at an angle θ to the downward vertical through O.

(a) Given that the tension in the string is 98 N, find θ and the angular speed of the particle.

(b) Given that the string breaks when the tension in it exceeds 196 N, find the greatest angular speed which the particle can have without the string breaking.

(AEB)

13. A rigid pole OA, of length $4l$, has the end O fixed to a horizontal table with the other end A vertically above O. The ends of a light inextensible string, of length $4l$, are fixed to A and a point B distance $2l$ below A on the pole. A small particle of mass m is fastened to the mid-point of the string and made to rotate, with both parts of the string taut, in a horizontal circle with angular speed ω. Find the tension in both parts of the string in terms of m, l, ω and the acceleration due to gravity, g.

At a given instant both parts of the string are cut. Assuming that there is no air resistance and that $\omega = \sqrt{(g / 3l)}$, find the time which elapses before the particle strikes the table and show that it does so at a point distant $3l$ from O.

(AEB)

8 SIMPLE HARMONIC MOTION

Objectives

After studying this chapter you should

- be able to model oscillations;
- be able to derive laws to describe oscillations;
- be able to use Hooke's Law;
- understand simple harmonic motion.

8.0 Introduction

One of the most common uses of oscillations has been in time-keeping purposes. In many modern clocks quartz is used for this purpose. However traditional clocks have made use of the pendulum. In this next section you will investigate how the motion of a pendulum depends on its physical characteristics.

The key feature of the motion is the time taken for one complete oscillation or swing of the pendulum. i.e. when the pendulum is again travelling in the same direction as the initial motion. The time taken for one complete oscillation is called the **period**.

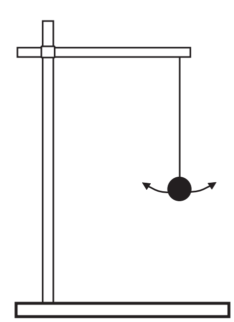
8.1 Pendulum experiments

Activity 1 Your intuitive ideas

To begin your investigation you will need to set up a simple pendulum as shown in the diagram. You will need to be able to

- vary the length of the string;
- vary the mass on the end of the string;
- record the time taken for a particular number of oscillations.

Once you are familiar with the apparatus try to decide which of the factors listed at the beginning of the next page affect the period. Do this without using the apparatus, but giving the answers that you intuitively expect.



The mass is attached by a string to the support, to form a simple pendulum.

(a) The length of the string

(b) The mass of the object on the end of the string.

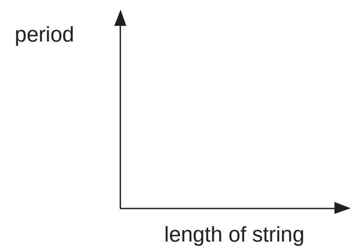
(c) The initial starting position of the mass.

Now try simple experiments to verify or disprove your intuitive ideas, using a table to record your results.

You are now in a position to start analysing the data obtained, using some of the basic mathematical concepts in pure mathematics.

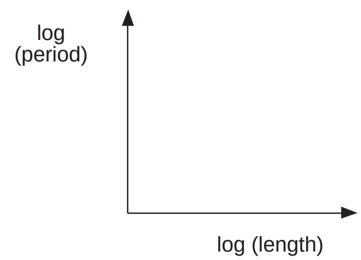
Activity 2 Analysis of results

You will probably have observed already that as you shortened the string the period decreased. Now you can begin to investigate further the relationship between the length of the string and the period.



- Plot a graph for your results, showing period against length of string.
- Describe as fully as possible how the period varies with the length of the string.

You may know from your knowledge of pure mathematics how a straight line can be obtained from your results using a log-log plot.



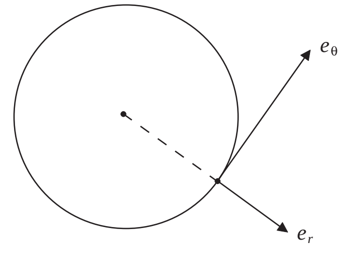
Log-log graph

- Plot a graph of log (period) against log (length of string), and draw a line of best fit.
- Find the equation of the line you obtain and hence find the relationship between the period and the length of the string.

8.2 Pendulum theory

You will have observed from Section 8.1 that the period of the motion of a simple pendulum is approximately proportional to the square root of the length of the string. In this section you are presented with a theoretical approach to the problem

The path of the mass is clearly an arc of a circle and so the results from Chapter 7, Circular Motion, will be of use here. It is convenient to use the unit vectors, $\mathbf{e}_r$ and $\mathbf{e}_\theta$, directed outwards along the radius and along the tangent respectively. The acceleration, **a**, of an object in circular motion is now given by



The unit vectors for circular motion

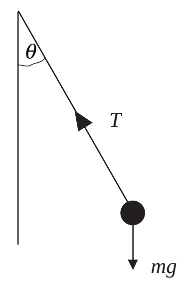
$$\mathbf{a} = -r\dot{\theta}^2\mathbf{e}_r + r\ddot{\theta}\mathbf{e}_\theta$$

where $$\dot{\theta} = \frac{d\theta}{dt} \text{ and } \ddot{\theta} = \frac{d^2\theta}{dt^2}.$$

Forces acting on the pendulum

As in all mechanics problems, the first step you must take is to identify the **forces** acting. In this case there is the tension in the string and the force of gravity. There will, of course, also be air resistance, but you should assume that this is negligible in this case. The forces acting and their resultant are summarised in the table below.

Force	Component form
T	$-T\mathbf{e}_r$
mg	$mg\cos\theta\,\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta$
Resultant	$(mg\cos\theta - T)\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta$



Now it is possible to apply Newton's second law, using the expression for the acceleration of an object in circular motion

$$\mathbf{F} = m\,\mathbf{a},$$

giving

$$(mg\cos\theta - T)\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta = m\left(-l\dot{\theta}^2\,\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta\right)$$

where $r = l$, the length of the string.

Equating the coefficients of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ in this equation leads to

$$T - mg\cos\theta = ml\dot{\theta}^2 \qquad (1)$$

and

$$-mg\sin\theta = ml\ddot{\theta}. \qquad (2)$$

From equation (2) $\ddot{\theta} = -\dfrac{g\sin\theta}{l}$.

If the size of θ is small, then $\sin\theta$ can be approximated by θ, so that

$$\boxed{\ddot{\theta} = -\frac{g\theta}{l}} \qquad (3)$$

Activity 3   Solving the equation

Verify that

$$\theta = A\cos\left(\sqrt{\frac{g}{l}}\,t + \alpha\right)$$

is a solution of equation (3), where α is an arbitrary constant.

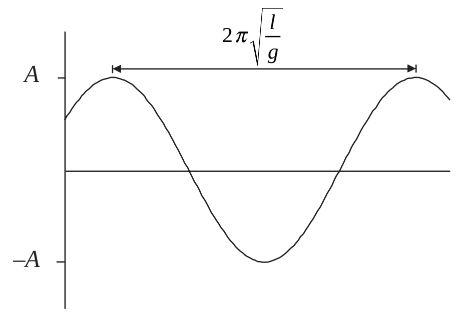
Interpreting the solution

Each part of the solution

$$\theta = A\cos\left(\sqrt{\frac{g}{l}}\,t + \alpha\right)$$

describes some aspect of the motion of the pendulum.

- The variable, A, is known as the **amplitude** of the oscillation. In this case the value of A is equal to the greatest angle that the string makes with the vertical.



- $\sqrt{\frac{g}{l}}$ determines how long it takes for one complete oscillation.

  When $\sqrt{\frac{g}{l}}\,t = 2\pi$ or $t = 2\pi\sqrt{\frac{l}{g}}$

  then the pendulum has completed its first oscillation. This time is known as the **period** of the motion.

  In general, if you have motion that can be described by

  $$x = A\cos(\omega t + \alpha)$$

  then the period, P, is given by

  $$P = \frac{2\pi}{\omega}.$$

  It is sometimes also useful to talk about the **frequency** of an oscillation. This is defined as the number of oscillations per second.

- The constant α is called the **angle of phase**, or simply the **phase**, and its value depends on the way in which the pendulum is set in motion. If it is released from rest the angle of phase will be zero, but if it is flicked in some way, the angle of phase will have a non-zero value.

Exercise 8A

1. A D.I.Y magazine claims that a clock that is fast (i.e. gaining time) can be slowed down by sticking a small lump of Blu-tac to the back of the pendulum. Comment on this procedure.
2. A clock manufacturer wishes to produce a clock, operated by a pendulum. It has been decided that a pendulum of length 15 cm will fit well into an available clock casing. Find the period of this pendulum.
3. The result obtained for the simple pendulum used the fact that $\sin\theta$ is approximately equal to θ for small θ. For what range of values of θ is this a good approximation? How does this affect the pendulum physically?
4. A clock regulated by a pendulum gains 10 minutes every day. How should the pendulum be altered to correct the time-keeping of the clock?
5. Two identical simple pendulums are set into motion. One is released from rest and the other with a push, both from the same initial position. How do the amplitude and period of the subsequent motions compare?
6. The pendulums in Question 5 have strings of length 20 cm and masses of 20 grams. Find equations for their displacement from the vertical if they were initially at an angle of $5°$ to the vertical and the one that was pushed was given an initial velocity of 0.5ms^{-1}.

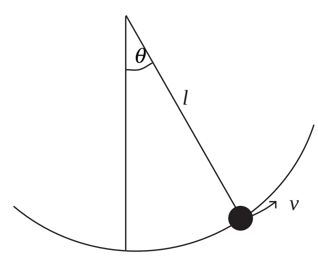
8.3 Energy consideration

An alternative approach is to use energy consideration.

As the simple pendulum moves there is an interchange of kinetic and potential energy. At the extremeties of the swing there is zero kinetic energy and the maximum potential energy. At the lowest point of the swing the bob has its maximum kinetic energy and its minimum potential energy.

Activity 4 Energy

For the simple pendulum shown opposite, find an expression for the height of the pendulum bob in terms of the angle θ. You may assume that you are measuring from the lowest point.



Explain why the potential energy of the bob is given by

$$mgl(1-\cos\theta).$$

The kinetic energy of the pendulum bob is given by $\frac{1}{2}mv^2$.

As no energy is lost from the system, the sum of potential and kinetic energies will always be constant, giving

$$T = mgl(1-\cos\theta)+\frac{1}{2}mv^2$$

where T is the total energy of the system. The value of T can be found by considering the way in which the pendulum is set into motion.

Finding the speed

Solving the equation for v gives

$$v = \sqrt{\frac{2T}{m} - 2gl(1 - \cos\theta)}.$$

This allows you to calculate the speed at any position of the pendulum.

You can also find an expression for θ from the equation for v.

Activity 5 Finding θ

Explain why the speed of the pendulum bob is given by

$$v = l\frac{d\theta}{dt}.$$

(You may need to refer to the Chapter 7, Circular Motion if you find this difficult.)

Substitute this into the equation for v and show that

$$\frac{d\theta}{dt} = \sqrt{\frac{2T}{ml^2} - \frac{2g}{l}(1 - \cos\theta)}.$$

Simplifying the equation

If you assume that the oscillations of the pendulum are small, then you can use an approximation for $\cos\theta$.

Activity 6 Small θ approximation

Use the approximation

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

to show that $\frac{d\theta}{dt}$ can be written in the form

$$\frac{d\theta}{dt} = \omega\sqrt{a^2 - \theta^2}, \text{ where } a \text{ is a constant.}$$

Integrating the equation

You can solve this equation by separating the variables to give

$$\int \frac{1}{\sqrt{a^2-\theta^2}}\,d\theta = \int \omega\,dt.$$

The LHS is simply a standard integral that you can find in your tables book and the RHS is the integral of a constant.

Activity 7 Integrating

Show that

$$\theta = \alpha\sin(\omega t + c)$$

and explain why this could be written as

$$\theta = \alpha\cos(\omega t + \alpha)$$

where α and ω are defined as above and c is an unknown constant given by $\alpha = c + \frac{\pi}{2}$.

This result is identical to that obtained earlier.

Exercise 8B

1. A pendulum consists of a string of length 30 cm and a bob of mass 50 grams. It is released from rest at an angle of $10°$ to the vertical. Draw graphs to show how its potential and kinetic energy varies with θ. Find the maximum speed of the bob.

2. Find expressions for the maximum speed that can be reached by a pendulum if it is set in motion at an angle $\theta°$ to the vertical if
   (a) it is at rest;
   (b) it has an initial speed u.

8.4 Modelling oscillations

In this section you will investigate other quantities which change with time, can be modelled as oscillations and can be described by an equation in the form $x = a\cos(\omega t + \alpha)$.

There are many other quantities that involve motion that can be described using this equation.

Some examples are the heights of tides, the motion of the needle in a sewing machine and the motion of the pistons in a car's engine. In some cases the motion fits exactly the form given above, but in others it is a good approximation.

Activity 8

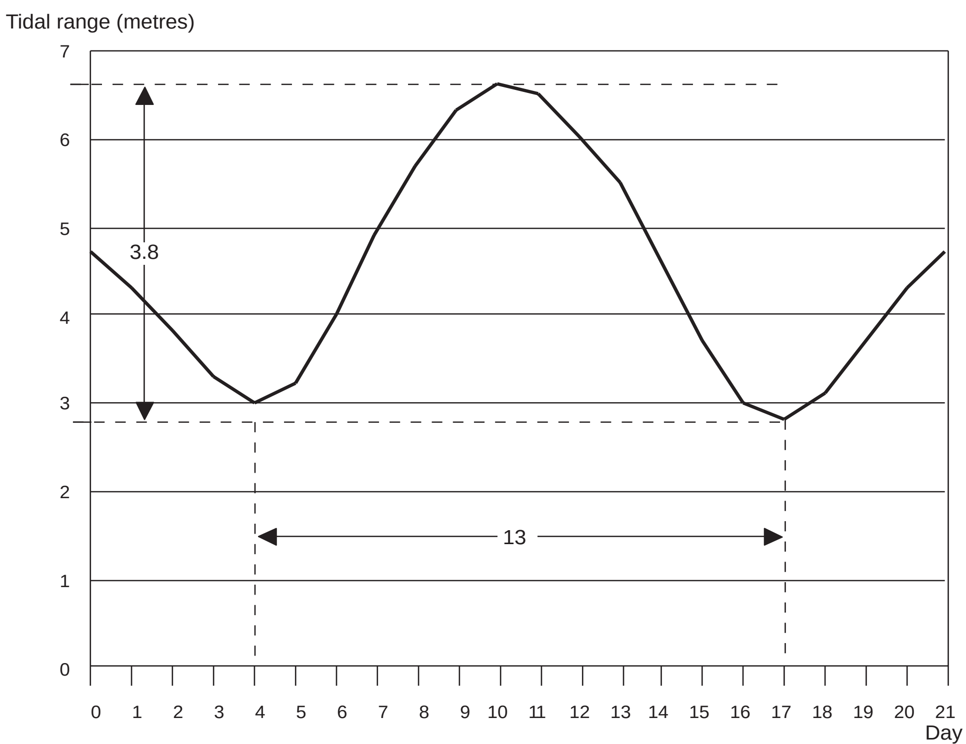
Find other examples of motion that can be modelled using the equation $x = a\cos(\omega t + \alpha)$.

Fitting the equation to data

One example that could be modelled as an oscillation using the equation is the range of a tide (i.e. the difference between high and low tides). The table shows this range for a three-week period.

Day	Range	Day	Range	Day	Range
1	4.3	8	5.7	15	3.7
2	3.8	9	6.3	16	3.0
3	3.3	10	6.6	17	2.8
4	3.0	11	6.5	18	3.1
5	3.2	12	6.0	19	3.7
6	4.0	13	5.5	20	4.3
7	4.9	14	4.6	21	4.7

The graph below shows range against day.



This graph is clearly one that could be modelled fairly well as an oscillation using $x = a\cos(\omega t + \alpha)$. One difference you will observe between this graph and those for the simple pendulum is that this one is not symmetrical about the time axis. The curve has, in fact, been translated upwards, so the range will be described by an equation of the form

$$R = a\cos(\omega t + \alpha) + b.$$

The difference between the maximum and minimum range is approximately 3.8 m. The amplitude of the oscillation will be half this value, 1.9 m. So the value of *a* in the equation will be 1.9.

As the graph is symmetrical about the line $r = 4.7$ the value of *b* will be 4.7.

It is also possible from the graph to see that the period is 13 days. From Section 8.2 you will recall that the period is given by

$$P = \frac{2\pi}{\omega}$$

so that $$\omega = \frac{2\pi}{P}.$$

In this case, $\omega = \frac{2\pi}{13}$, which is in radians.

This leaves the value of α to be determined. The equation is now

$$R = 1.9\cos\left(\frac{2\pi t}{13} + \alpha\right) + 4.7.$$

When $t = 1$, $R = 4.3$, so using these values

$$4.3 = 1.9\cos\left(\frac{2\pi}{13} + \alpha\right) + 4.7$$

$$\Rightarrow \quad 1.9\cos\left(\frac{2\pi}{13} + \alpha\right) = -0.4$$

$$\Rightarrow \quad \cos\left(\frac{2\pi}{13} + \alpha\right) = -\frac{0.4}{1.9}$$

$$\Rightarrow \quad \frac{2\pi}{13} + \alpha = 1.78$$

$$\Rightarrow \quad \alpha = 1.78 - \frac{2\pi}{13}$$

$$\alpha \approx 1.30$$

which gives

$$R = 1.9\cos\left(\frac{2\pi t}{13} + 1.3\right) + 4.7$$

(Note that in applying this result you must use radians.)

Activity 9

Draw your own graph of the data on tidal ranges.

Superimpose on it the graph of the model developed above.

Compare and comment.

Exercise 8C

1. In the UK 240 volt alternating current with a frequency of 50 Hz is utilised. Describe the voltage at an instant of time mathematically.
2. The tip of the needle of a sewing machine moves up and down 2 cm. The maximum speed that it reaches is 4 ms^{-1}. Find an equation to describe its motion.
3. Black and Decker BD538SE jigsaws operate at between 800 and 3200 strokes per minute, the tip of the blade moving 17 mm from the top to the bottom of the stroke. Find the range of maximum speeds for the blade.
4. A person's blood pressure varies between a maximum (systolic) and a minimum (diastolic) pressures. For an average person these pressures are 120 mb and 70 mb respectively. Given that blood pressure can be modelled as an oscillation, find a mathematical model to describe the changes in blood pressure. It takes 1.05 seconds for the blood pressure to complete one oscillation.
5. The motion of the fore and hind wings of a locust can be modelled approximately using the ideas of oscillations. The motion of the fore wings is modelled by

   $$F = 1.5 + 0.5\sin(1.05t - 0.005)$$

   where F is the angle between the fore wing and the vertical. Find the period and the amplitude of the motion.
   Each hind wing initially makes an angle of $1.5°$ to the vertical. It then oscillates with period 0.06 s and amplitude $1.5°$. Form a model of the form

   $$h = H + a\sin(kt)$$

   for the motion of one of the hind wings.

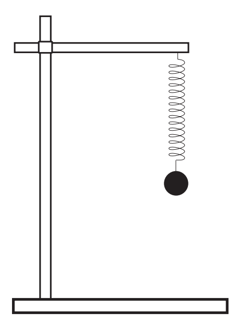
8.5 Springs and oscillations

In Section 6.5, Hooke's Law was used as the model that is universally accepted for describing the relationship between the tension and extension of a spring. Hooke's Law states that

$$T = ke$$

where T is the tension, k the spring stiffness constant and e the extension of the spring. If the force is measured in Newtons and the extension in metres, then k will have units Nm^{-1}.

To begin your investigations of the oscillations of a mass/spring system you will need to set up the apparatus as shown in the diagram opposite.



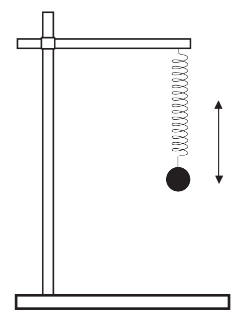
Masses can be attached to the spring suspended from the stand

The equipment you will need is

- a stand to support your springs
- a variety of masses
- 2 or 3 identical springs
- a ruler
- a stopwatch.

Activity 10   Intuitive ideas

If you pull down the mass a little and then release it, it will oscillate, up and down. Once you are familiar with the apparatus, try to decide how the factors listed below affect the period. Do this without using the apparatus, giving the answers that you intuitively expect.



(a) The mass attached to the spring.

(b) The stiffness of the spring.

(c) The initial displacement of the mass.

Also consider some simple experiments to verify or disprove your intuitive ideas.

Theoretical analysis

To begin your theoretical analysis you need to identify the resultant force.

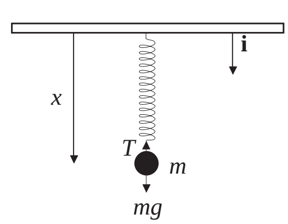
Explain why the resultant force is $(mg - T)\mathbf{i}$

Activity 11

Use Hooke's Law to express T in terms of k, l and x, where k is the stiffness of the spring, l the natural length of the spring and x the length of the spring.

So the resultant force on the mass is

$$(mg - kx + kl)\mathbf{i}$$



and the acceleration of the mass will be given by $\frac{d^2x}{dt^2}\mathbf{i}$

Activity 12

Use Newton's second law to obtain the equation

$$\frac{d^2x}{dt^2}+\frac{k}{m}x=g+\frac{kl}{m}$$

and show that

$$x=a\cos\left(\sqrt{\frac{k}{m}}t+\alpha\right)+\frac{mg}{k}+l$$

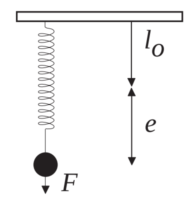
satisfies this equation.

Why is the period of these oscillations given by

$$P=2\pi\sqrt{\frac{m}{k}}\ ?$$

Work done

Consider a spring of stiffness k and natural length l_0. If a force, F, is applied to cause the spring to stretch, then this force must increase as the spring extends. So the **work done** in stretching a spring is evaluated using



Work done in stretching a spring

$$\begin{aligned}\text{Work done } &= \int_0^e F\,dx\\ &= \int_0^e kx\,dx\\ &= \left[\frac{1}{2}kx^2\right]_0^e\\ &= \frac{1}{2}ke^2.\end{aligned}$$

So the energy stored in a spring is given by $\frac{1}{2}ke^2$. When the spring is released the energy is converted into either kinetic or potential energy.

Simple harmonic motion

If the equation describing the motion of an object is of the form

$$x=a\cos(\omega t+\alpha)+c$$

then that type of motion is described as **simple harmonic motion** (SHM).

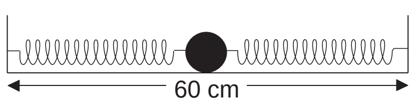
This equation is deduced from the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = b$$

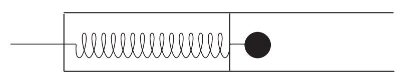
Both the simple pendulum and the mass/spring system are examples of SHM.

Exercise 8D

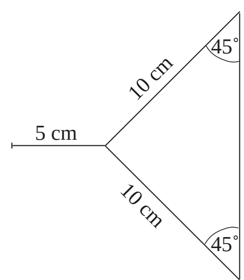
1. A clock manufacturer uses a spring of stiffness $40\ \text{Nm}^{-1}$. It is required that the spring should complete four oscillations every second. What size mass should be attached to the spring? What initial displacement would be required?
2. A 250 gram mass is attached to a spring of natural length 40 cm and stiffness $200\ \text{Nm}^{-1}$. The mass is pulled down 3 cm below its equilibrium position and released. Find
   (a) an expression for its position;
   (b) the period of its motion;
   (c) the amplitude of its motion.
3. A baby bouncer is designed for a baby of average mass 18 kg. The length of the elastic string cannot exceed 80 cm, in order to ensure flexibility of use. Ideally the bouncer should vibrate at a frequency of 0.25 Hz. Determine the stiffness constant of the elastic string.
4. Two identical springs are used to support identical masses. One is pulled down 3 cm from its equilibrium position. The other is pulled down 2 cm from its equilibrium position. How do the amplitude and period of the resulting motion compare?
5. Two identical springs are attached to the 2 kg mass that rests on a smooth surface as shown.



   The springs have stiffness $30\ \text{Nm}^{-1}$ and natural lengths 25 cm. The mass is displaced 2 cm to the left and then released. Find the period and amplitude of the resulting motion.
6. A mass/spring system oscillates with period 0.07 s on earth. How would its period compare if it were moved to the moon?
7. The spring in a pinball machine is pulled back with a plunger and then released to fire the balls forward. Assume that the spring and the ball move in a horizontal plane.



   The spring has stiffness $600\ \text{Nm}^{-1}$ compressed by 5 cm to fire the ball. The mass of the ball is 50 grams. Find its speed when it leaves contact with the plunger.
8. A catapult is arranged horizontally. It is made from an elastic string of stiffness $80\ \text{Nm}^{-1}$. The diagram below shows the initial dimensions of the catapult, before the elastic is stretched. The stone is placed in the catapult and pulled back 5 cm.



   Find the work done in pulling the catapult back and the speed of the stone when it leaves the catapult. The stone has a mass of 25 grams.

   Find the speed of the stone if the catapult is arranged to fire the stone vertically rather than horizontally.
9. Turbulence causes an aeroplane to experience an up and down motion that is approximately simple harmonic motion. The frequency of the motion is 0.4 Hz and the amplitude of the motion is 1 m. Find the maximum acceleration of the aeroplane.

10. A buoy of mass 20 kg and height 2 m is floating in the sea. The buoy experiences an upward force of $400d$, where d is the depth of the bottom of the buoy below the surface.

    (a) Find the equilibrium position of the buoy.

    (b) Find the period and amplitude of the motion of the buoy if it is pushed down 0.3 m from its equilibrium position.

11. A metal strip is clamped at one end. Its tip vibrates with frequency 40 Hz and amplitude 8 mm. Find the maximum values of the magnitude of the acceleration and velocity.

8.6 Miscellaneous Exercises

1. A seconds pendulum is such that it takes one second for the pendulum to swing from one end of its path to the other end, i.e. each half of the oscillation takes 1 second.

   (a) Find the length of the seconds pendulum.

   (b) A seconds pendulum is found to gain one minute per day. Find the necessary change in length of the pendulum if the pendulum is to be made accurate.

2. A simple pendulum oscillates with period $2t$ seconds. By what percentage should the pendulum length be shortened so that it has a period of t seconds?

3. A particle is moving in simple harmonic motion and has a speed of 4 ms^{-1} when it is 1 m from the centre of the oscillation. If the amplitude is 3 m, find the period of the oscillation.

4. A light elastic string with a natural length of 2.5 m and modulus 15 N, is stretched between two points, P and Q, which are 3 m apart on a smooth horizontal table. A particle of mass 3 kg is attached to the mid-point of the string. The particle is pulled 8 cm towards Q, and then released. Show that the particle moves with simple harmonic motion and find the speed of the particle when it is 155 cm from P.

5. A particle describes simple harmonic motion about a point O as centre and the amplitude of the motion is a metres. Given that the period of the motion is $\frac{\pi}{4}$ seconds and that the maximum speed of the particle is 16 ms^{-1}, find

   (a) the speed of the particle at a point B, a distance $\frac{1}{2}a$ from O;

   (b) the time taken to travel directly from O to B. (AEB)

6. A particle performs simple harmonic motion about a point O on a straight line. The period of motion is 8 s and the maximum distance of the particle from O is 1.2 m. Find its maximum speed and also its speed when it is 0.6 m from O.

   Given that the particle is 0.6 m from O after one second of its motion and moving away from O, find how far it has travelled during this one second. (AEB)

7. A particle describes simple harmonic motion about a centre O. When at a distance of 5 cm from O its speed is 24 cm s^{-1} and when at a distance of 12 cm from O its speed is 10 cm s^{-1}. Find the period of the motion and the amplitude of the oscillation.
   Determine the time in seconds, to two decimal places, for the particle to travel a distance of 3 cm from O. (AEB)

8. A particle moves along the x-axis and describes simple harmonic motion of period 16 s about the origin O as centre. At time $t = 4$ s, $x = 12$ cm and the particle is moving towards O with speed $\frac{5\pi}{8} \text{ cm s}^{-1}$. Given that the displacement, x, at any time, t, may be written as

   $$x = a\cos(\omega t + \phi),$$

   find a, ω and ϕ. (AEB)

9. A particle, P, of mass m is attached, at the point C, to two light elastic strings AC and BC. The other ends of the strings are attached to two fixed points, A and B, on a smooth horizontal table, where $\text{AB} = 4a$. Both of the strings have the same natural length, a, and the same modulus. When the particle is in its equilibrium position the tension in each string is mg. Show that when the particle performs oscillations along the line AB in which neither string slackens, the motion is simple harmonic with period $\pi\sqrt{\left(\frac{2a}{g}\right)}$.

   The breaking tension of each string has magnitude $\frac{3mg}{2}$. Show that when the particle is performing complete simple harmonic oscillations the amplitude of the motion must be less than $\frac{1}{2}a$.

   Given that the amplitude of the simple harmonic oscillations is $\frac{1}{4}a$, find the maximum speed of the particle. (AEB)

10. The three points O, B, C lie in that order, on a straight line l on a smooth horizontal plane with $\text{OB} = 0.3$ m, $\text{OC} = 0.4$ m.

    A particle, P, describes simple harmonic motion with centre O along the line l. At B the speed of the particle is 12 ms^{-1} and at C its speed is 9 ms^{-1}. Find

    (a) the amplitude of the motion;

    (b) the period of the motion;

    (c) the maximum speed of P;

    (d) the time to travel from O to C.

    This simple harmonic motion is caused by a light elastic spring attached to P. The other end of the spring is fixed at a point A on l where A is on the opposite side of O to B and C, and $\text{AO} = 2$ m. Given that P has mass 0.2 kg, find the modulus of the spring and the energy stored in it when $\text{AP} = 2.4$ m. (AEB)

11. A particle, P, describes simple harmonic motion in the horizontal line ACB, where C is the mid-point of AB. P is at instantaneous rest at the points A and B and has a speed of 5 ms^{-1} when it passes through C. Given that in one second P completes three oscillations from A and back to A, find the distance AB. Also find the distance of P from C when the magnitude of the acceleration of P is $9\pi^2$ ms^{-2}.

    Show that the speed of P when it passes through the point D, which is the mid-point of AC, is $\dfrac{5\sqrt{3}}{2}$ ms^{-1}. Also find the time taken for P to travel directly from D to A. (AEB)

12. A particle moves with simple harmonic motion along a straight line. At a certain instant it is 9 m away from the centre, O, of its motion and has a speed of 6 ms^{-1} and an acceleration of $\dfrac{9}{4}$ ms^{-2}.

    Find

    (a) the period of the motion;

    (b) the amplitude of the motion;

    (c) the greatest speed of the particle.

    Given that at time $t = 0$, the particle is 7.5 m from O and is moving towards O, find its displacement from O at any subsequent time, t, and also find the time when it first passes through O. (AEB)

9 PHYSICAL STRUCTURES

Objectives

After studying this chapter you should

- be able to analyse the forces and turning effects on simple structures and frameworks;
- be able to find the centre of mass of simple and composite bodies;
- be able to analyse the stability and toppling conditions of a simple body.

9.0 Introduction

The objects in the list opposite may not at first seem to have much in common but they are all examples of structures. They either hold up or support something, reach across or span a distance or contain or protect something. They all, however, in some way support a load, so a structure can be thought of as an assembly of materials so arranged that loads can be supported.

a chair and occupant;
a dam;
the Severn Road Bridge;
a cable-stayed roof;
a ballet dancer balancing on one foot;
a snow-laden tree;
a spider's web with a fly trapped in it.

The chair both holds you up and supports your weight. The dam contains and supports the water behind it. The bridge spans the River Severn and supports both its own weight and that of the traffic. The roof protects the inside of the building and has its weight supported. The ballet dancer's weight is supported by one foot. The tree trunk supports the branches as well as the extra load of snow. The spider's web contains and supports the fly.

The weight of the load the structure supports needs to be transferred to the ground in some way. The chair supports your weight but the ground supports both your weight and the chair's, this load being transferred to the ground via the chair legs.

Your body's skeleton is a structure. What are its purposes?

Activity 1 Investigating structures

What are the purposes of the structures listed opposite?

What are the loads on them?

How are these loads supported?

a tower crane;
a suspension bridge;
a spider's web;
a diving board;
a North Sea oil platform;
a wheelbarrow;
a trellis with creeper.

As the examples in Activity 1 show, structures are very widespread. Some are natural, some man-made. They are made from a variety of materials: stone, wood, iron, steel, brick, reinforced concrete, bone and tissue being common ones.

The book of Genesis in the Bible describes an early structure of brick, the Tower of Babel. The pyramids in Egypt, the Parthenon in Athens, St Peter's in Rome, the Taj Mahal in India, great cathedrals like Durham, York and Salisbury and castles like Stirling and Harlech, the Iron Bridge in Shropshire, are all structures built many years ago and still standing today. Structures like these satisfy the basic requirement that they must not break or fail.

Not all structures succeed in this. Trees can break under the weight of snow on them. Bridges such as the old Tay Railway Bridge in Scotland and the Tacoma Narrows Bridge in the United States have collapsed. The roofs of medieval cathedrals did sometimes fall in. Babylonian houses collapsed and the Code of Hammarabic in about 2000 BC made the builder pay with his life if the collapse killed the owner.

Structures fail when a part breaks or is permanently disturbed in some way. This can happen because the forces on the structure become too great for it to bear. The forces may be due to the weight of whatever the structure is carrying or supporting or to its own weight. They may also be due to external forces such as wind and snow or the impact of some object on the structure. Wind causes the Eiffel Tower in Paris to sway up to 12.7 cm and trees are blown over in strong gales. Cars in collision are damaged.

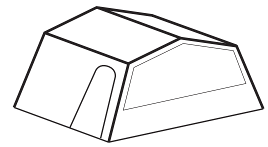
Since it is important that a structure achieves its purpose and does not fail, man-made structures need to be designed carefully. However, the designer needs to take account not only of safety but also of the cost and method of construction and the appearance of the structure. A bridge built from the opposite banks of a river must meet in the middle. A skyscraper is out of place in the countryside.

Mathematics plays an important part in the design of structures, especially in their mechanics. The aim of this chapter is to find out something of how mathematics is applied via mechanics in the design of structures and how it can also be used to help understand why man-made and natural structures are the way they are.

9.1 Introducing frameworks

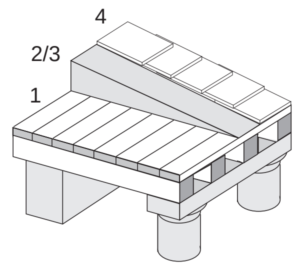
A frame tent consists of a frame and a covering. The frame is usually made from aluminium poles which are strong but extremely lightweight. The complete frame is very light compared to the loads it supports. The load is not simply the force on the frame due to the weight of the covering, but also the forces due to the wind and the rain to which the tent might be subjected.

A framework is a structure which consists of a system of connected members.



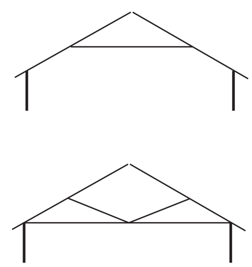
In the case of the frame tent, the system of connected members is the frame of poles. A framework is designed to support a weight or load, and generally it is very light compared to the load it supports.

A second example is the roof truss, the framework which supports a roof. In fact, roof trusses have not always been used to put pitched roofs on to buildings. The following method was used by the ancient Greeks to build their temples:-



1. Put a flat timber roof over the walls and a large number of supporting pillars.
2. Heap thousands of tonnes of earth on to this.
3. Shape the earth to the desired pitch.
4. Lay tiles directly on to this earth.

This crude but interesting method was superseded by the development of the roof truss.



The A-shaped roof truss is a very simple example of a framework. It is light, yet rigid and strong enough to support the roof covering. Several A-shaped roof trusses spanning the walls of a building could support a thatched or tiled roof on horizontally attached laths. The modern roof truss is very little altered from its early counterpart, although a considerable variety of designs are now used.

Not all frameworks are man-made. There are plenty of examples of natural frameworks such as animals' skeletons and spiders' webs.

Activity 2   Finding frameworks

Discuss and list other examples of frameworks.

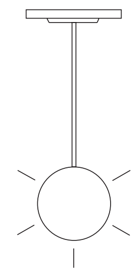
Look for small examples such as shelf brackets as well as larger ones such as electricity pylons.

Try also to find buildings in which frameworks have been made into features rather than hidden. Naturally occurring frameworks are less obvious, but try to find some of these also.

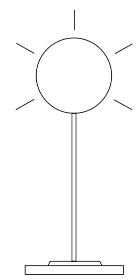
Many of the frameworks you find in Activity 2 are likely to be three dimensional as are the frame tent and skeleton examples. In this section you analyse only two dimensional frameworks but should be aware that three dimensional ones are probably more common.

Ties and struts

The frameworks you have considered so far each consist of rigid members joined at their ends to form a structure which does not collapse or move. The members used are often metal bars or wooden struts. It is important to know how these behave when the framework is loaded.



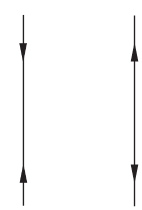
The first diagram opposite shows a light fitting suspended from the ceiling by a metal rod. It could be replaced by something flexible like a chain or string and still support the light fitting. The force in the rod supporting the light fitting is a **tension**.



The second diagram shows the same light fitting made into a standard lamp. The weight of the light fitting now tends to compress the rod. In this case it could not be replaced by something flexible and still support the light fitting. The force in the rod supporting the light fitting in this case is a **thrust**.

When a rod is in **tension** it is said to be a **tie**.

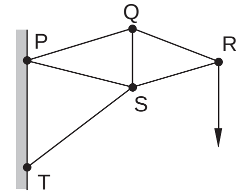
When a rod is in **thrust** it is said to be a **strut**.



When any framework is subjected to a load, some members will be in tension and act as ties; others will be in thrust and act as struts.

Example

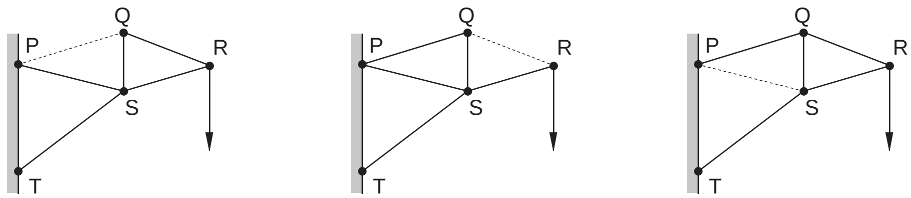
For the framework PQRST, say whether the light rods are ties or struts.



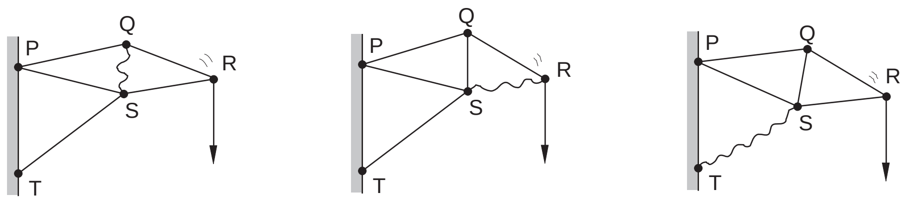
Solution

A good strategy is to consider each of the rods in the framework removed in turn.

If the rod could be successfully replaced by string, then it is in tension, a tie. If its replacement by string would result in collapse of the framework, then the rod is in thrust, a strut.



The rods PQ, QR and PS could all be successfully replaced by string.

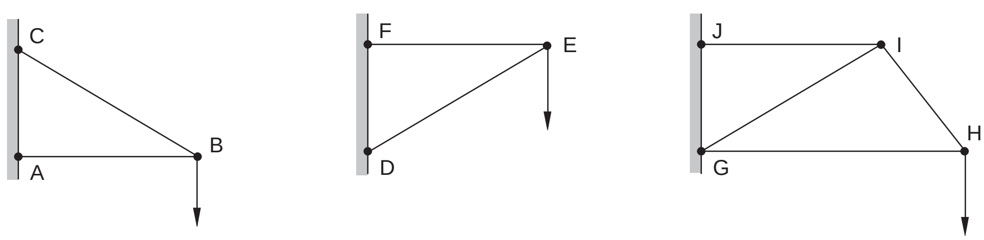


The rods QS, SR and ST could not be successfully replaced by string.

PQ, QR and PS are ties. QS, SR and ST are struts.

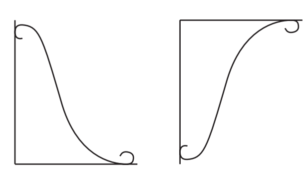
Activity 3 Ties and struts

For these simple frameworks, say whether the light rods are ties or struts.

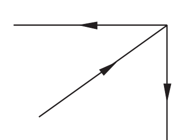


Compare the two simple frameworks of hanging basket brackets shown opposite. Discuss how they differ in terms of which members are ties and which are struts.

Look for other simple frameworks to analyse in the same way.



Even the simplest of frameworks often have joints at which there are three or more rods connected.

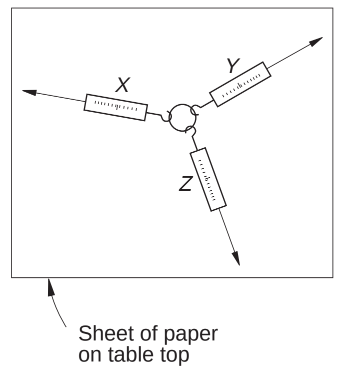


Forces in equilibrium

When a framework does not collapse or move supporting a load, then the forces in the rods acting at its joints must be in equilibrium. If a framework is to be designed to support a load, it is necessary to find the forces which act in its members. This is known as **analysing the framework**. In order to analyse a framework, you need to be able to solve problems concerning the equilibrium of forces. One condition for a system of forces to be in equilibrium is that their resultant is zero. This idea was first discussed in Section 4.5 and is now explored. Often for frameworks the forces act at a point.

Activity 4 Equilibrium of forces at a point

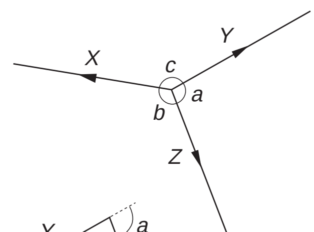
You will need 3 newton meters, a metal ring (for instance, a key ring) and paper.



Hook each newton meter into the metal ring as shown opposite and pull on the end of each.

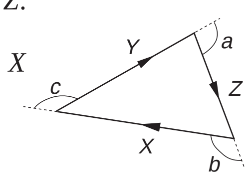
When the ring is in equilibrium, that is, it does not move, mark the positions of the centre of the ring and the handle end of each newton meter.

Note also the readings *X*, *Y* and *Z* on the newton meters.



Remove the newton meters and use the marks on the paper to find the angles *a*, *b* and *c* between the forces *X*, *Y* and *Z*.

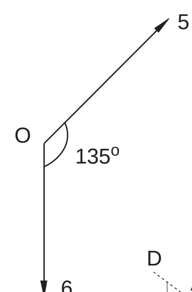
Draw a scale diagram of the forces *X* followed by *Y* followed by *Z*.



What do you notice about the positions of the beginning of force *X* and end of force *Z* on your scale diagram?

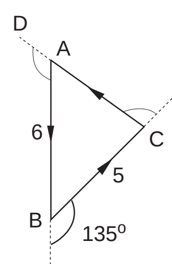
Can you explain the result in terms of the triangle of forces?

Example



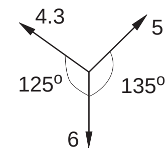
Two forces, in newtons, act at the point O as shown in the diagram opposite. Find, by scale diagram, the magnitude and direction of the third force acting at O which maintains equilibrium.

Solution



Choose a scale so that the diagram fits comfortably on the page; 1 cm for 1 N is satisfactory in this case. Draw the two given forces, one followed by the other.

The third force at O required to maintain equilibrium can be measured on your scale drawing and is 4.3 N at $125°$ to the 6 N force.



There are a number of alternative methods of solving problems such as those in the last example.

You know how to resolve a force into two components at right angles using the parallelogram law (see Sections 4.2 and 4.3).

This can be used as the basis of one alternative method for solving problems involving the equilibrium of forces acting at a point. The last example shown is reconsidered using this method.

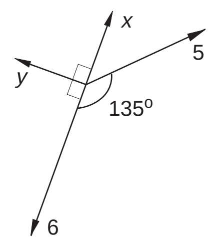
Alternative solution

Let the required force have components *X* and *Y* opposite to and at right angles to the 6 N force.

The sum of the forces in the direction of *X* is zero,

giving

$$X + 5\cos 45° - 6 = 0$$
$$X = 6 - 5(0.707)$$
$$X = 2.46.$$



The sum of the forces in the direction of *Y* is zero,

giving

$$Y - 5\cos 45° = 0$$
$$Y = 5(0.707)$$
$$Y = 3.54.$$

The required force is then

$$\sqrt{X^2 + Y^2}$$
$$= 4.31 \text{ N}.$$

Its angle to the 6 N force is

$$180 - \tan^{-1}\left(\frac{Y}{X}\right)$$
$$= 180 - \tan^{-1}(1.43)$$
$$= 180 - 55.1$$
$$= 124.9°$$

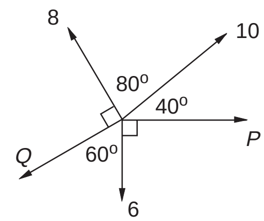
The technique used in the above example relies upon forces in equilibrium at a point having zero resultant. It then follows that:

If forces are in equilibrium at a point, then the sum of the components of the forces in any one direction is zero.

By using this result in two directions at right angles, it is possible to set up two equations and solve for two unknowns.

Example

The forces *P*, *Q*, 6, 8 and 10 N in the diagram opposite are in equilibrium. Find forces *P* and *Q*.

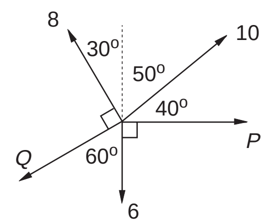


Solution

The sum of the forces in the direction at right angles to *P* is zero, giving

$$6 + Q\cos 60° - 10\cos 50° - 8\cos 30° = 0.$$

This is known as **resolving** in the direction at right angles to *P*. The reason for starting with this direction is that it leads to an equation in *Q* only. It simplifies to

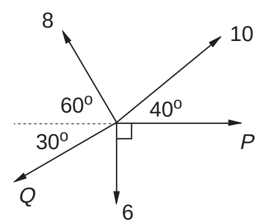


$$0.5Q = 10\cos 50° + 8\cos 30° - 6$$

giving $0.5Q = 7.36$

so that $Q = 14.72$.

The sum of the forces in the direction of *P* is zero and so resolving in this direction gives



$$P + 10\cos 40° - Q\cos 30° - 8\cos 60° = 0.$$

Substituting the value of *Q* gives

$$P = 14.72\cos 30° + 8\cos 60° - 10\cos 40°$$

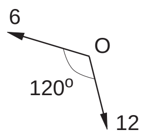
or $P = 9.08.$

The forces are $P = 9.08$ N and $Q = 14.72$ N.

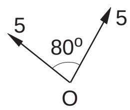
Exercise 9A

1. In each case, find by scale diagram the force at O required to maintain equilibrium. Forces are in newtons.

(a)

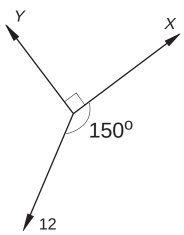


(b)

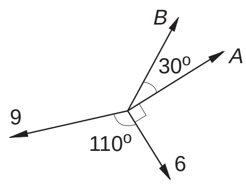


2. If each set of forces is in equilibrium, find the unknown forces. Forces are in newtons.

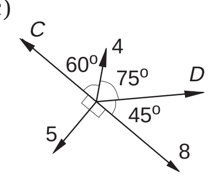
(a)



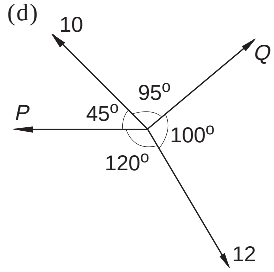
(b)



(c)



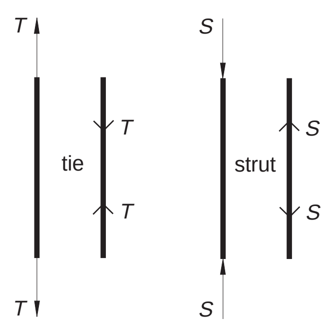
(d)



9.2 Analysis of a simple framework

The analysis of a simple framework involves finding the forces acting in each of its members when the framework is subject to a load. This would then allow a suitable material and cross-section to be chosen for each member so that it could withstand the calculated forces. So this analysis is part of the design process for the framework.

Some assumptions have to be made in the force analysis of a framework. Firstly, it is assumed that all members are two-force members. That is, each is in equilibrium under the action of two equal and opposite forces applied at its ends.



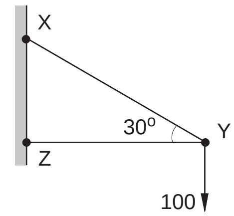
These two forces are tensions T in the case of a tie, or thrusts S, in the case of a strut. Equal and opposite forces to those applied at its ends act within the tie or strut to maintain equilibrium.

Secondly, it is assumed that the weight of the framework is so small compared to the load it supports that this weight can be neglected. For individual members of the framework, this means that the weight of a member is so small that it can be neglected compared to the force it supports.

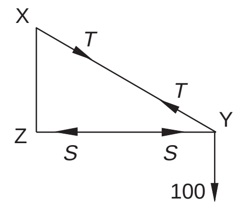
Thirdly, it is assumed that whether the members of the framework are bolted, riveted or welded to each other at joints, these joints are points in equilibrium due to the action of the forces in those members. It is for this reason that the method of resolving at a point can be used to find the forces in the members of a framework.

Example

For the simple framework XYZ supporting a load of 100 N at Y, find the forces in its members XY and YZ. Find also the forces exerted by the framework on the wall at X and Z.

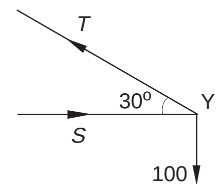


Solution



By inspection, there is a tension, T, in XY and a thrust, S, in YZ.

Since the point Y is in equilibrium under the action of the three forces T, S and 100 N, resolving vertically gives



$$T\cos 60° - 100 = 0$$

so that

$$0.5T = 100$$

$$T = 200$$

and resolving horizontally gives

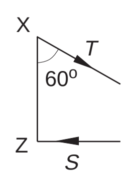
$$S - T\cos 30° = 0$$

so that

$$S = 200(0.866)$$

$$S = 173.$$

The member XY is a tie supporting a tension of 200 N



and YZ is a strut supporting a thrust of 173 N.

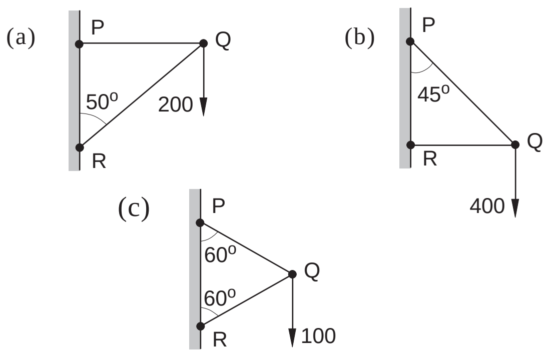
The force exerted by the framework on the wall at X is T, that is 200 N acting away from the wall at $60°$ to the downward vertical.

The force exerted by the framework on the wall at Z is S, that is 173 N acting towards the wall at $90°$ to it.

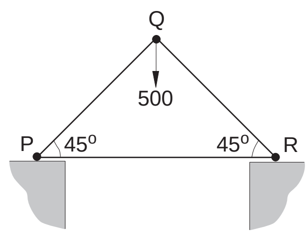
Note: If the directions for S and T are chosen incorrectly in the first part of this solution, it does not matter. The values of S and T would work out negative from the equations indicating that the directions are opposite to those chosen.

Exercise 9B

1. Find the forces in the members PQ and QR in each case and the forces exerted by the framework on the wall at P and R. Forces are in newtons.

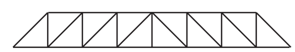


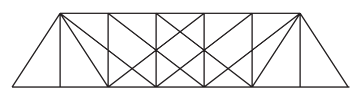
2. Framework PQR rests on the supports P and R as shown. It is not fixed. Find the forces in the members PQ, QR and PR and the normal contact forces exerted by the supports at P and R.

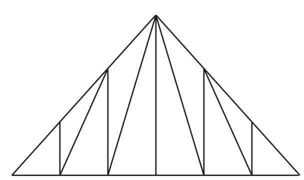


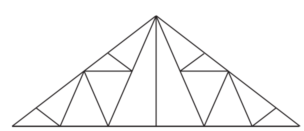
9.3 More complex frameworks

More complex frameworks are often found in bridge trusses as well as in roof trusses. They are often named after the engineers who first designed them. When the railroads pushed westwards in North America, wooden truss bridges were used to span the wide rivers of the West and a considerable number of American engineers invented trusses. Two bridge trusses and two roof trusses are shown on the right.


Howe


Whipple - Murphey


Pratt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Activity 5 Bridge and roof trusses

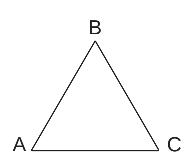
You will find bridge trusses on railway and foot bridges. Many modern buildings have their roof trusses painted as a feature, rather than hidden.

Find and sketch some roof and bridge trusses different from the four named ones.

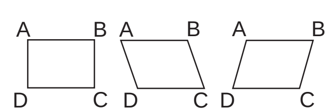
See if you can also research the names of the extra trusses you find.

Perfect trusses

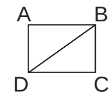
You should realise that a triangle of rods is a rigid structure, whereas a square is not.



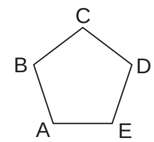
ABC cannot move.



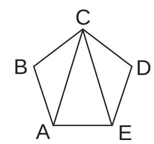
If one extra rod is added to ABCD it becomes rigid as shown opposite.



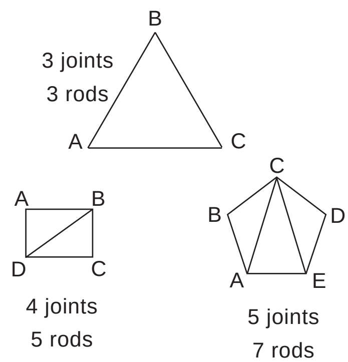
A pentagon of rods can move.



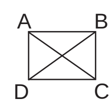
If two extra rods are added to ABCDE, it becomes rigid as shown opposite.



The three examples shown opposite are of **correctly triangulated trusses**. They have just enough rods so that they are rigid. They are also known as **perfect trusses**.



If two extra rods had been added to the square, it would have been rigid, but one rod would have been unnecessary. It would not have been a perfect truss.



Activity 6   The perfect truss

You can use geostrips and connectors to try out the ideas in this activity, or simply treat it as a pencil and paper activity.

Investigate the least number of rods necessary for a perfect truss of 3, 4, 5, 6, 7 ..., joints.

Can you generalise the result and find a formula for the least number of rods necessary for a perfect truss of n joints?

Look at the four named trusses and others you have sketched. Are they perfect trusses?

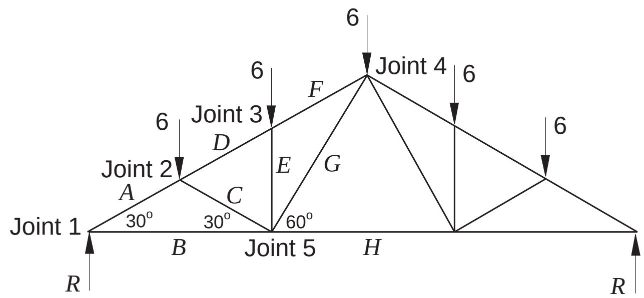
Suggest some reasons why some of the roof and bridge trusses you have found are not perfect trusses.

9.4 Method of joints

The method used to analyse frameworks in Exercise 9B is a simple case of the method of joints. This method can be used to analyse the forces in more complicated frameworks.

Example

Find the forces in all the members of the roof truss shown in the diagram.



The load due to the roof takes the form of five separate 6 kN forces acting vertically as shown. The truss is supported by two vertical contact forces.

Solution

The framework and the loads acting upon it are symmetric, so the contact forces are equal, to R say, and only the joints on one side of the line of symmetry need to be considered.

The complete roof truss is in equilibrium and so resolving vertically gives

$$2R - 30 = 0$$

so that $R = 15.$

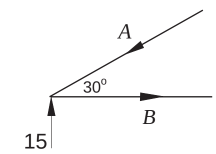
Since the framework is in equilibrium, the forces acting at each joint are in equilibrium. Joints 1, 2, 3, 4 and 5 are taken in turn to find the forces A, B, C, D, E, F, G and H in the members as shown.

The directions of A and B are reasonably obvious here and so are inserted as shown in the first diagram on the next page.

If the direction of a force is not obvious, simply choose a direction and find its value. If it turns out to be negative, you know the direction is opposite to that chosen.

Joint 1

Resolving vertically gives



$$A\sin 30° = 15$$

so that $A = 30.$

Resolving horizontally gives

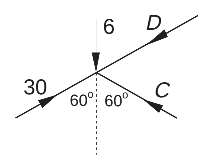
$$B = \mathrm{A}\cos 30°$$

so $B = 30(0.866)$

and $B = 26.0.$

Joint 2

Resolving vertically gives



$$6 + D\cos 60° = 30\cos 60° + C\cos 60°$$

so $12 + D = 30 + C$

or $D - C = 18.$ (1)

Resolving horizontally gives

$$30\cos 30° = D\cos 30° + C\cos 30°$$

so $D + C = 30.$ (2)

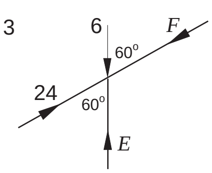
Solving equations (1) and (2) gives

$$D = 24$$

and $C = 6.$

Joint 3

Resolving vertically gives



$$6 + F\cos 60° = E + 24\cos 60°$$

so $12 + F = 2E + 24$

and $F - 2E = 12.$ (3)

Resolving horizontally gives

$$24\cos 30° = F\cos 30°$$

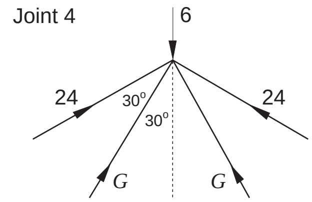
so $F = 24$

and from equation (3)

$$E = 6.$$

Joint 4

The direction of G is not obvious but is inserted as shown.



Resolving vertically gives

$$6 = 2(24\cos 60) + 2G\cos 30$$

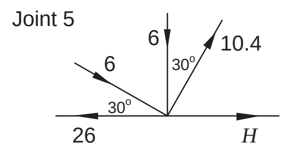
so $\quad 6 = 24 + 1.73G$

and $\quad G = -10.4.$

This shows that G is in the opposite direction to that on the diagram.

Joint 5

Resolving horizontally gives



$$H + 10.4\cos 60 + 6\cos 30 = 26$$

so $\quad H + 5.2 + 5.2 = 26$

and $\quad H = 15.6.$

So the forces in the members are:

$$A = 30,\quad B = -26.0,\quad C = 6,\quad D = 24,\quad E = 6,$$
$$F = 24,\quad G = -10.4 \text{ and } H = 15.6, \text{ all in kN.}$$

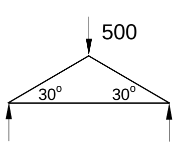
Negative signs indicate tensions in ties.
All others are thrusts in struts.

Exercise 9C

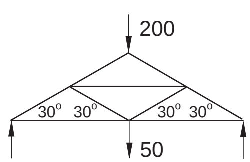
In Questions 1 to 4, calculate the force in each rod of the framework together with any vertical contact forces indicated. All forces are in newtons and each framework can be assumed to be in a vertical plane.

Questions 3 and 4 represent parts of a truss as used in some bridges.

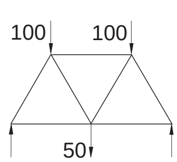
1.



2.

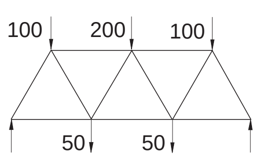


3.



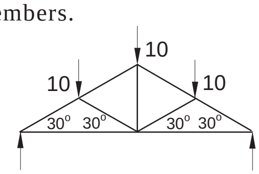
rods all of equal length

4.



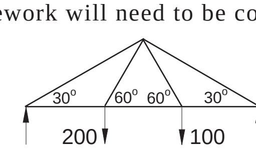
rods all of equal length

5. The roof truss in the diagram below supports a roof which can be considered to act as three separate loads of 10 kN as shown. Find the two vertical contact forces and the forces in all the members.



6. The framework shown below is subject to the loads 200 N and 100 N. Two vertical contact forces support the framework. Find the contact forces and the force in each member of the framework.

   (Hint: Although the framework is symmetric, it is not loaded symmetrically. All the jointsin the framework will need to be considered.)



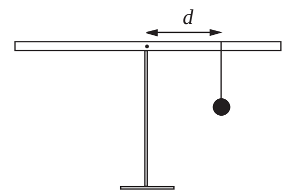
9.5 Lifting devices

The list opposite gives examples of lifting devices, some of which you meet in this section.

a see-saw;
a wheelbarrow being pushed;
a pair of scissors;
a fishing rod;
the arm of a shot putter;
a discus thrower;
a road barrier;
a crane.

Activity 7 Balancing weights

You need a stand, a metre rule with a hole at 50 cm, Blu-tack, masses (3×10 g, 50 g, 100 g) and cotton to suspend masses.



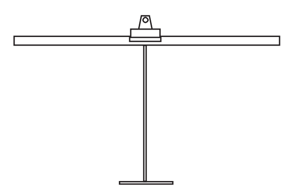
Pivot the rule at its centre on the stand, and if necessary use Blu-tack to balance it.

Suspend one mass on the left hand side of the pivot and two masses on the right hand side, so that the rule balances.

Use your data to deduce a rule which relates weights of the masses and distances from the pivot.

Verify your rule by suspending different combinations of masses on either side of the pivot.

Note the weights and the distances of their points of suspension from the pivot.



[If a metre rule with a hole in it is not available, use a bull-dog clip and stand. Take measurements from the centre of the rule. This is the point referred to as the pivot in the text.]

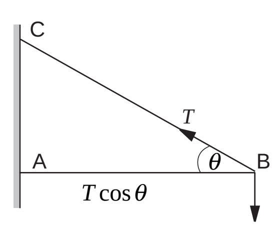
You should have found that the number

$$\text{weight} \times \text{distance from the pivot}$$

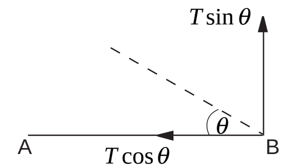
is important in this balance problem. This is called the **moment** of the weight about the pivot and measures the turning effect the weight produces about that point. Weights on opposite sides of the pivot balance if their moments about the pivot are equal.

9.6 Moment of a force

In Activity 7 the line from the pivot to the point of suspension of each mass is perpendicular to the direction in which its weight is acting. In the framework shown opposite, the weight suspended at B is acting vertically downwards and so its direction is perpendicular to the rod AB. The moment about A is (weight) $\times$ AB. The tension, T, in the tie BC acts at an angle, θ, to AB. What is its moment about A?



The tension, T, can be resolved into its components $T\sin\theta$ perpendicular to AB and $T\cos\theta$ along BA. Since the component $T\cos\theta$ acts along BA, it does not have any turning effect about A. The component $T\sin\theta$ does, however, have a turning effect about A. It is measured by its moment



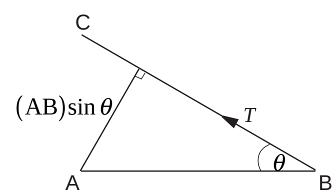
$$(T\sin\theta)\times(\text{AB}).$$

The earlier definition is therefore modified to

Moment of a force acting at point B about point A	=	(component of force perpendicular to AB) × (distance AB)

Since

$$(T\sin\theta)\times\text{AB} = T\times(\text{AB}\sin\theta)$$
$$= T\times\left(\begin{array}{c}\text{perpendicular distance from A to}\\ \text{the line BC along which } T \text{ acts}\end{array}\right),$$



an alternative definition of moment is

The moment of a force acting at point B about point A is equal to the force times the **perpendicular** distance from A to the line along which the force acts.

Since

$$\text{moment} = (\text{force}) \times (\text{distance})$$

moments are measured in newton metres (Nm).

In Activity 7 two other forces act: the weight of the rule and the normal contact force at the pivot.

Why do these forces not come into the earlier calculations?

The weight of the rule acts through its midpoint. You can check this by balancing the rule about its midpoint on your finger. The only forces on the rule are its weight and the force on it due to your finger. Since the force due to your finger acts at the midpoint, for these forces to balance, the weight must also act there and so its moment about the midpoint, that is the pivot, is zero. The normal contact force also acts at the pivot. Its moment about that point is zero and so neither of these forces contributes to the earlier calculations.

The point in a body through which the weight acts is called the **centre of gravity**. This idea is developed further in Section 9.10. For now, note that

The centre of gravity of a uniform rod is at its midpoint.

Moments arise in many situations. When you open a door you apply a force to the knob or handle and there is a turning effect and so a moment about the hinge. If you have two spanners, one longer than the other, and you apply the same force to both, you achieve a larger turning effect with the longer spanner. Some of you may have been boating on the canals and had to manoeuvre lock gates. Here you usually push against the gates so that you are perpendicular to the bars and walk in a circular path. In this way you achieve the maximum turning effect.

In the human body through muscle contraction, forces are applied at points where tendons are attached to the bones and moments arise about the joints.

To put a screw into a piece of wood you turn the screwdriver clockwise, whereas to take it out you turn the screwdriver anticlockwise. This suggests that moments are either in a clockwise or an anticlockwise sense.

Moments are taken to be **positive** when in the **anticlockwise** sense and **negative** when in the **clockwise** sense.

In each case the force, *F*, has a turning effect about a line through P, perpendicular to the plane of the paper. The direction of this line is associated with the moment to give a vector quantity.

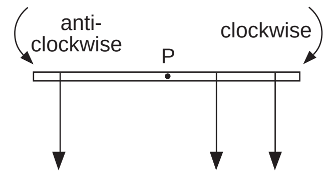
Activity 8   Moments

It is obvious why the door handle is as far as possible from the hinge and which line the door turns about.

List other situations in which moments arise.

Identify the line about which there is a turning effect.

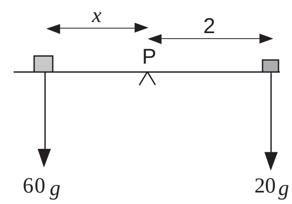
Principle of moments



In Activity 7 you found that the rule only balances when the moments of the weights suspended on one side of the pivot are equal to those on the other side. The weights on the right hand side of P have negative moment whereas those on the left hand side have positive moment and so an equivalent statement is that the ruler balances at P when the algebraic sum of the moments of the weights about P is zero. This is an example of the **principle of moments**, one form of which is

> For a body in equilibrium under a system of forces acting at different points in the body, the algebraic sum of moments of the forces about **any** point is zero.

Example



A see-saw pivoted at its centre rests in a horizontal position. A boy of mass 20 kg sits on it 2 m from the pivot. How far from the pivot should his father of mass 60 kg sit if the see-saw is to balance?

Solution

The boy has a negative moment about P

$$= -20g \times 2 = -40g.$$

His father has a positive moment about P

$$= 60g \times x = 60gx.$$

By the principle of moments

$$-40g + 60gx = 0,$$

so that

$$40g = 60gx$$

giving

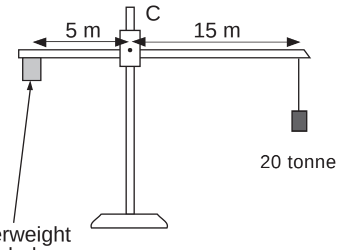
$$x = \frac{40}{60} = \frac{2}{3} \text{ m.}$$

What other forces act on the see-saw? Why are they not considered?

Example

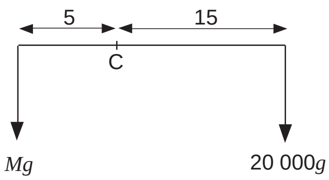
In the tower crane shown opposite, the distances from the cabin, C, of the suspended mass of 20 tonne and the counterweight are as shown. Assuming that the mass of the crane can be neglected compared with the suspended mass and the counterweight, determine the mass of the counterweight necessary for the crane to be in equilibrium. If the suspended mass is now trebled, determine where it must be positioned for the crane to be in equilibrium with the same counterweight.

[1 tonne $\equiv$ 1000 kg]



Solution

If the counterweight has mass M, applying the principle of moments about C gives



$$-15\times 20\times 1000g + 5Mg = 0,$$

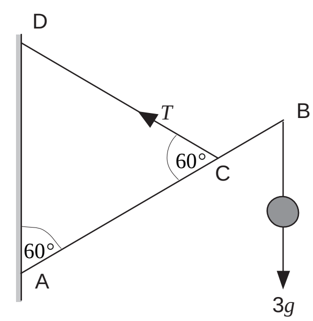
so that

$$M = 60\times 10^3 \text{ kg}$$
$$= 60 \text{ t.}$$

If the suspended mass is now 60 t, the same as the counterweight, for equilibrium it must be positioned 5 m to the right of the cabin for the crane to be in equilibrium.

Example

A 40 cm long rod, AB, has a 3 kg mass hanging from B. It is hinged at A and is supported by a chain CD. If the masses of the rod and chain can be neglected compared with the suspended mass, find the tension in the chain if ACD is an equilateral triangle of side 30 cm and D is vertically above A.



Solution

In Section 9.2 you solved this type of problem using resolution of forces. Here is an alternative method using the principle of moments.

The forces acting on the rod are the tension, T, in the chain, the suspended weight and the force exerted by the hinge at A. Applying the principle of moments at A, the force at A does not come into the calculation since it has zero moment about A. Neither T nor the suspended weight is perpendicular to the rod, but the components of these two forces perpendicular to the rod are $T\cos 30$ and $3g\cos 30$ respectively.

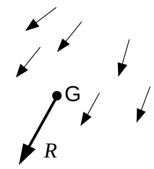
Applying the principle of moments about A gives

$$T\cos 30 \times 30 - 3g\cos 30° \times 40 = 0,$$

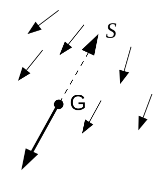
so that

$$T = 4g = 40\,\text{N},$$

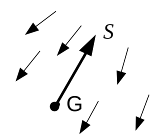
putting $g = 10\ \text{ms}^{-2}$.



A useful alternative form of the principle of moments is described in the next unit.

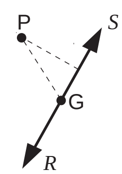


Suppose that the resultant of the system of forces has magnitude *R* and acts at G. Now at G introduce a force *S* which has magnitude *R* and acts in the opposite direction to the resultant. The force *S* cancels out the resultant force and so the original system of forces together with *S* is in equilibrium. By the earlier statement, the algebraic sum of the moments of all these forces about any point P is zero, so that



$$(\text{moment of } S \text{ about P}) + \left(\begin{array}{c}\text{algebraic sum of the moments of the}\\ \text{original system of forces about P}\end{array}\right) = 0$$

But



$$(\text{moment of } S \text{ about P}) = -(\text{moment of } R \text{ about P})$$

and so, combining these two statements gives

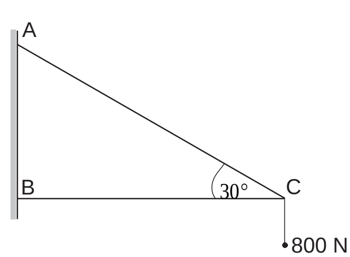
$$\text{moment of } R \text{ about P} = \begin{array}{c}\text{moments of the original}\\ \text{system of forces about P.}\end{array}$$

> The moment of the resultant of a system of forces about any point is equal to the algebraic sum of the moments of the forces about that point.

Exercise 9D

1. A see-saw of length 4 m, pivoted at its centre, rests in a horizontal position. John, who weighs 30 kg, sits on one end. Where should his friend James, who weighs 40 kg, sit if the see-saw is to balance?

2. The diagram shows two thin rods of negligible weight jointed together at C and anchored to a vertical wall at A and B. Rod BC is horizontal and rod AC makes an angle of $30°$ to the horizontal. A load of 800 N hangs from the joint at C.

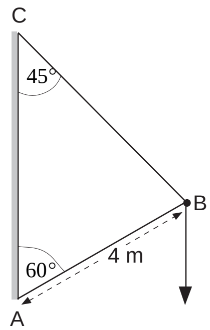


(a) Find the tension or thrust in each of the rods.

(b) Which of the rods could be replaced by a sufficiently strong length of cable without otherwise altering the structure or causing it to move?

(c) Determine the force exerted by the rod BC on the wall.

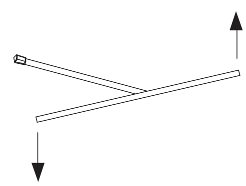
3. The figure shows a rod AB, whose weight can be neglected, hinged at A and connected to C by a cable BC. A mass of 4 tonnes is suspended from B. Calculate the tension in the cable BC. Find the thrust in the rod AB.



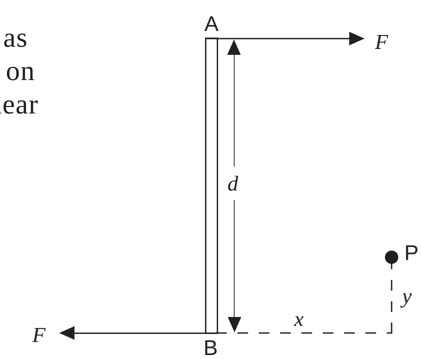
4. Repeat Question 3 with the mass suspended 3 m along the rod from A. Also determine the force exerted by the cable on the wall.

5. Susan and Alison, who weigh 30 kg and 36 kg, sit on a see-saw 1.8 m and 1.6 m respectively on the left of the pivot. The see-saw is pivoted at its centre, is 4 m long, and when unoccupied, rests in a horizontal position. Their father, who weighs 70 kg, sits to the right of the pivot.
   (a) Where should he sit for the see-saw to balance?
   (b) Does the father's position have to change significantly if the children change places with each other?
   (c) Can their father always balance the see-saw wherever the children sit on the left hand side?
   (d) Is this the case if their mother, who weighs 60 kg, changes places with their father?
   (e) The girls sit on the left of the pivot with Alison 1 m from the pivot and their mother on the opposite side. If their mother sits x m and Susan y m from the pivot, what is the relation between x and y for the see-saw to balance?

9.7 Couples



A special spanner is sometimes used to remove the nuts which hold on a car wheel. Force is applied to the arms of a T-shaped spanner and there is a turning effect on the nut; the longer the arms of the 'T', the greater the turning effect.



If two equal and parallel forces, F, act in opposite directions as shown on the diagram opposite, then there is a turning effect on the rod AB. Such a pair of forces is called a **couple**. The linear resultant in any direction is zero and there is no translational effect.

Taking moments about A gives

$$M = -Fd$$

and taking moments about B also gives

$$M = -Fd.$$

Furthermore, taking moments about P gives

$$\begin{aligned} M &= -F(d-y) - Fy \\ &= -Fd. \end{aligned}$$

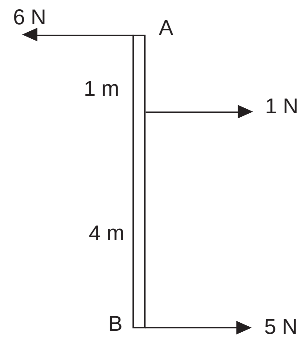
These three results show that the moment of the couple is not zero and is independent of the position of the point about which moments are taken.

To show that a couple exists, therefore, you need to establish that

- the linear resultant of the couple is zero
- the resultant moment of the couple is not zero.

Example

Show that the forces on the rod AB, shown opposite, form a couple and find the moment of this couple.



Solution

Resolving the forces perpendicular to AB gives

$$1+5-6=0,$$

thus there is no linear resultant of the forces and the forces form a couple since they do not act at a point.

Taking moments about A gives

$$\begin{aligned} M &= 1\times 1+5\times 5 \\ &= 26 \text{ Nm.} \end{aligned}$$

Alternatively, taking moments about B gives

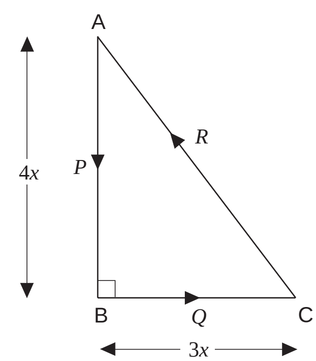
$$\begin{aligned} M &= 6\times 5-1\times 4 \\ &= 26 \text{ Nm.} \end{aligned}$$

The forces therefore form a couple with a resultant moment of 26 Nm anticlockwise.

The examples considered so far have systems of parallel forces but other frameworks with non parallel forces can be such that the forces form a couple.

Example

The diagram opposite shows a system of forces P, Q and R acting along the sides of a right-angled triangle ABC. Find the ratio $P : Q : R$ if the system of forces is equivalent to a couple.



Solution

By Pythagoras' theorem, $\text{AC} = 5x$.

If the system of forces is a couple, the linear resultant is zero.

Resolving vertically,

$$R\sin \text{A}\hat{\text{C}}\text{B} - P = 0$$

$$R\frac{4x}{5x} - P = 0$$

$$P = \frac{4R}{5}.$$

Resolving horizontally,

$$R\cos A\hat{C}B - Q = 0$$

$$R\frac{3x}{5x} - Q = 0$$

$$Q = \frac{3R}{5},$$

so $Q : R = 3 : 5$.

Also $P : Q = \frac{4R}{5} : \frac{3R}{5}$

$= 4 : 3,$

so $P : Q : R = 4 : 3 : 5$.

Exercise 9E

1. A rectangle PQRS has forces of 3 N acting along the sides PQ, QR, RS and SP respectively. If $PQ = 5$ m and $QR = 2$ m, show that a couple is acting and find the moment of the couple.
2. A regular hexagon ABCDEF with side length l has forces of $7a$, $6a$, $4a$, $4a$, $3a$ and a acting along the sides AB, CB, CD, DE, FE and FA respectively. Show that the forces are equivalent to a couple and find the magnitude of the couple.
3. Three forces given by the vectors $(-4\mathbf{i} + \mathbf{j})$, $(3\mathbf{i} - 2\mathbf{j})$ and $(\mathbf{i} + \mathbf{j})$ act through the points $(-\mathbf{i} + 4\mathbf{j})$, $(4\mathbf{i} - 2\mathbf{j})$ and $(2\mathbf{i} + 2\mathbf{j})$ respectively. Show that the forces are equivalent to a couple and find the moment of the couple.

9.8 Contact forces

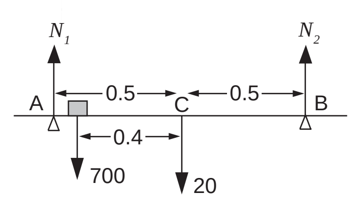
Before the next example, recall the following:

For a body to be in equilibrium under a system of forces acting at different points in the body:

1. the resultant force must be zero;
2. the algebraic sum of moments about **any** point is zero.

Example

A plank of length 2 m and mass 2 kg rests horizontally on smooth supports positioned 0.5 m either side of its centre. John, who weighs 70 kg, is standing on the plank 0.4 m from the centre. Find the forces exerted on the plank by the supports.



Solution

Let N_1 and N_2 be the normal contact forces (in newtons) exerted on the plank by the supports.

Since the plank is in equilibrium, the resultant force is zero, so that

$$N_1 + N_2 = 720.$$

The sum of moments about **any** point is also zero.

To simplify the calculation, take moments about either A or B as this gives an equation with only one unknown.

Taking moments about A gives

$$-700 \times (0.1) - 20 \times (0.5) + N_2 = 0$$

and so

$$N_2 = 70 + 10 = 80.$$

Also

$$N_1 = 720 - N_2 = 640.$$

The forces exerted by the supports at A and B are 640 N and 80 N respectively.

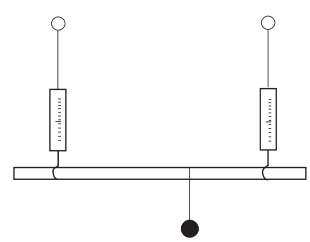
In this example, since all the forces are in the vertical direction, it is only necessary to consider forces in one direction and take moments about one point. In situations in which the forces do not act in one direction, it is necessary to consider the components of the forces in two directions and equal to zero. In many problems these directions may be horizontal and vertical but these may not always be the most appropriate directions.

Activity 9 Finding the contact forces

You need a metre rule, 2 newton meters, masses and some string loops.

Weigh the metre rule.

Place a known mass on the rule at a known distance from one end.



Attach the newton meters to the metre rule using loops of string.

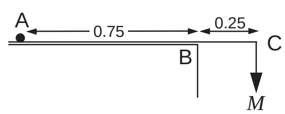
Holding the newton meters, read off the contact forces.

Calculate the theoretical values of the contact forces. Do these agree with your readings?

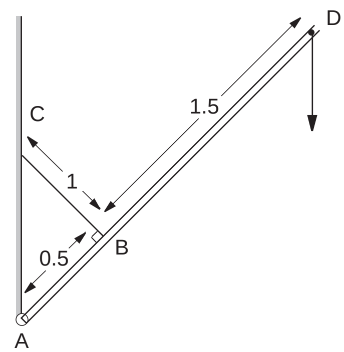
Repeat for different positions and with different masses.

Exercise 9F

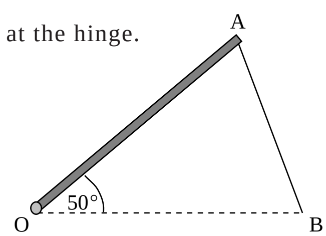
1. A 50 g mass and a 100 g mass are suspended from opposite ends of a metre rule which has mass 100 g. If the rule and the masses are attached to a string and hang vertically with the rule in a horizontal position, find
   (a) the tension in the string;
   (b) the position on the rule at which the string should be attached.
2. The rule and masses from Question 1 are now suspended by two strings instead of one. If the strings are attached 30 cms either side of the centre of gravity of the metre rule, find the tensions in the strings.
3. A plank of length 1.6 m and mass 4 kg rests on two supports which are 0.3 m from each end of the plank. A mass is attached to one end of the plank. If the normal contact force on the support nearest to this load is twice the normal contact force on the other support, determine the mass attached.
4. A metre rule is pivoted 20 cm from one end A and is balanced in a horizontal position by hanging a mass of 180 g at A. What is the mass of the rule? What additional mass should be hung from A if the pivot is moved 10 cm nearer to A?
5. A metre rule of mass 100 g is placed on the edge of a table as shown with a 200 g mass at A. A mass *M* grammes is attached at C.



   (a) When the rule is just on the point of overturning, where does the normal contact force act?
   (b) Determine the maximum value of *M* for which the rule will not overturn. Set up this experimental arrangement and test your predictions.
6. A plank of length 2 m and mass 6 kg is suspended in a horizontal position by two vertical ropes, one at each end. A 2 kg mass is placed on the plank at a variable point P. If either rope snaps when the tension in it exceeds 42 N, find the section of the plank in which P can be.
7. A uniform plank of length 2 m and mass 5 kg is connected to a vertical wall by a pin joint at A and a wire CB as shown. If a 10 kg mass is attached to D, find:
   (a) the tension in the wire;
   (b) the reaction at the pin joint A.



8. A loft door OA of weight 100 N is propped open at 50° to the horizontal by a strut AB. The door is hinged at O; $OA = OB = 1$ m. Assuming that the mass of the strut can be neglected compared to the mass of the door and that the weight of the door acts through the midpoint of OA, find:
   (a) the force in the strut;
   (b) the reaction at the hinge.



9.9 Stability

You may wonder what the items opposite have in common. All of them can be regarded as structures which must not collapse either by toppling or sliding.

dam; bridge; skyscraper; cathedral; gymnast; dog; insect; tree.

Not all structures succeed in this! (For example: *the Leaning Tower of Pisa; a collapsed bridge; an athlete fallen on the track*.)

Structures which do succeed are called **stable**, ones which do not, **unstable**.

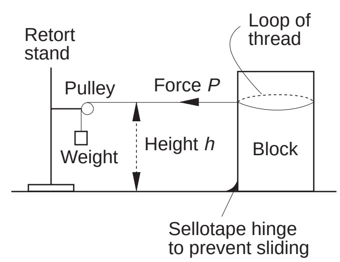
What causes a structure to become unstable?

All structures have external forces acting on them. For example, high winds produce significant forces on animals including human beings as well as on tall buildings. Dams are acted on by the forces due to the water they hold back. All structures are acted on by the Earth's gravitational field, both through their own weight and the weight of whatever load they carry. When these forces are too large or act in the wrong places, the structure falls. Structures need to be designed to withstand the loads they are likely to encounter though exceptionally strong forces can still cause failure. Some structures, such as a hurdle on an athletics track, are deliberately designed to fail when the force on them exceeds a certain amount.

High winds can produce such large forces on people that they are blown over. A practical simulation of this uses a rectangular block or box.

Activity 10 Toppling a tower

You need a block of wood, thread, pulley, retort stand, masses, Sellotape and sandpaper.



Vary the height, *h*, at which the thread is looped round the block by varying the height of the pulley. (The retort stand and clamp are useful here.)

For different *h* find the force, *P*, which just causes one end of the block to lift off the ground. Investigate the relation between *P* and *h*.

Why does the block not lift off the ground for smaller values of P?

Now remove the Sellotape and, for different heights, *h*, investigate whether the block topples or slides as *P* is increased from zero.

Place a sheet of sandpaper under the block and investigate how this affects it sliding or toppling.

Keep your data for use in Activity 14. Weigh the block since this weight is also needed.

Why do trees not topple over in high winds? What does happen to them?

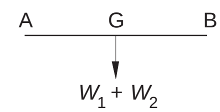
9.10 The centre of gravity of a body

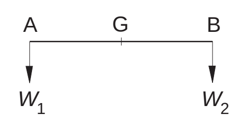
In Section 9.6 you met the centre of gravity of a rule, the fixed point of the rule through which its weight acts. All bodies and systems of particles have centres of gravity and these are important in stability.

The centre of gravity of two particles

The weights W_1 and W_2 of two particles at A and B have a resultant $W_1 + W_2$ parallel to W_1 and W_2. The line along which this resultant acts meets the line AB at a point G whose position can be found using the principle of moments.

From the second form of the principle the moment of the resultant $W_1 + W_2$ about any point is the sum of the moments of W_1 and W_2 about that point. Since the moment of the resultant about G is zero, the sum of moments of W_1 and W_2 about G is zero, so that G is a convenient point about which to take moments. This gives





$$W_1 \times (\mathrm{AG}) - W_2 \times (\mathrm{BG}) = 0$$

or

$$W_1 \times (\mathrm{AG}) - W_2 \times (\mathrm{AB} - \mathrm{AG}) = 0,$$

$$W_1 \mathrm{AG} + W_2 \mathrm{AG} - W_2 AB = 0$$

$$\mathrm{AG}(W_1 + W_2) = W_2 \mathrm{AB}$$

so that

$$\mathrm{AG} = \frac{W_2}{W_1 + W_2} \times (\mathrm{AB}).$$

The weights W_1 and W_2 of the two particles at A and B are equivalent to a single weight $W_1 + W_2$ at G. The point G is the centre of gravity of the two particles.

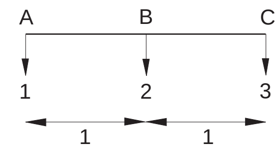
For particles of equal weight $\mathrm{AG} = \frac{1}{2}\mathrm{AB}$ and so the centre of gravity is at the midpoint of AB.

The centre of gravity of any number of particles on the same horizontal straight line can be found using the principle of moments.

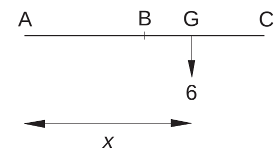
Example

Three particles A, B, C of weights 1 N, 2 N, 3 N respectively, lie on the same horizontal line as shown with $AB = BC = 1$ m. Find the distance of their centre of gravity, G, from A.

Solution



The resultant of the three weights is a weight 6 N acting through G.



The point G is again a convenient point about which to take moments since the moment of the resultant about G is zero and so the sum of the moments of the three weights about G is zero.

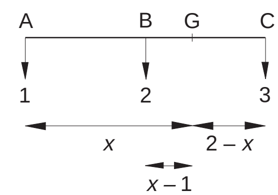
This gives

$$1\times(AG)+2\times(BG)-3\times(CG)=0,$$

or, with $AG = x$,

$$1.x+2(x-1)-3(2-x)=0,$$

so that $\quad 6x-8=0$



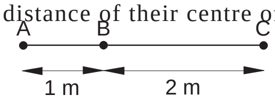
or $\quad x=\dfrac{4}{3}.$

This gives $\quad AG = 1.33$ m.

Exercise 9G

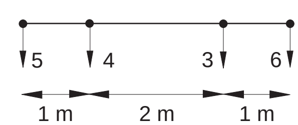
1. Find the centres of gravity of two particles
   (a) of weights 3 N and 5 N a distance of 1 m apart;
   (b) of weights 4 N and 7 N a distance of 2 m apart;
   (c) of weights 5 N and 10 N a distance of 9 m apart.

2. Three particles A, B,C, of weights 5 N, 3 N, 4 N, respectively, lie on the same horizontal line as shown. Find the distance of their centre of gravity from A.



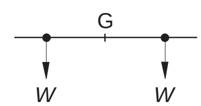
   The particle A is removed and replaced by a new particle. What is the greatest value of its weight if the centre of gravity of the three particles is to lie in BC?

3. Four particles of weights 5 N, 4 N, 3 N and 6 N lie on the same horizontal line as shown. How far is their centre of gravity from the 5 N particle?



9.11 The centres of gravity of some simple bodies

The result that the centre of gravity of two particles of equal weight is at the midpoint of the line joining the particles can be used to find the centre of gravity of simple bodies.



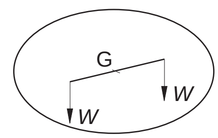
Since a uniform rod, such as a rule, is symmetric about its midpoint, G, it can be regarded as made up of pairs of particles of equal weight equidistant from G. Since the centre of gravity of each pair of particles is at G, the centre of gravity of the rod is at G.

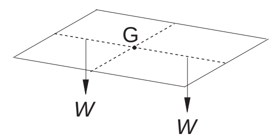
The same argument shows that:

the centre of gravity of a uniform circular disc is at its centre;

the centre of gravity of a rectangle is at the point where the lines forming the midpoints of opposite sides intersect;

the centre of gravity of a rectangular block is at the point where the diagonals forming opposite vertices intersect.



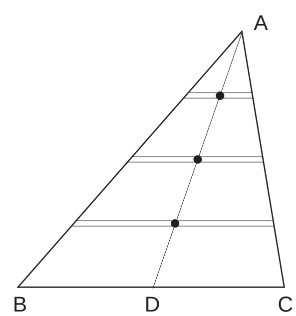


In each case and generally

The weight of the body can be regarded as acting at its centre of gravity.

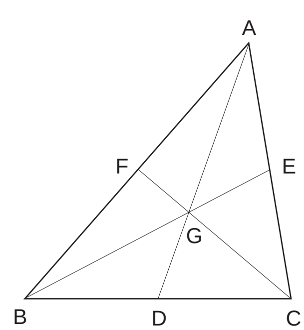
The centre of gravity of a triangle

A triangle can be thought of as made up of rods parallel to one of its sides BC. Since the centre of gravity of each rod is at its midpoint, the centre of gravity G of the triangle lies on the line joining A to the midpoint D of BC, the median AD.



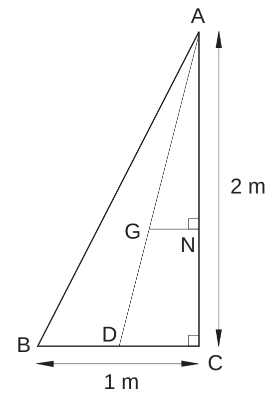
Repeating this argument for the other two sides shows that G is at the point of intersection of the medians AD, BE and CF, where E and F are the midpoints of AC and AB. From geometry

$$AG = \frac{2}{3}AD, \quad BG = \frac{2}{3}BE, \quad CG = \frac{2}{3}CF.$$



Example

Find the distances of the centre of gravity G from BC and AC in the triangle shown opposite.



Solution

Triangles AGN and ADC are similar, so

$$\frac{AN}{AC} = \frac{GN}{DC} = \frac{AG}{AD} = \frac{2}{3},$$

which gives $AN = \frac{2}{3}AC$ and $GN = \frac{2}{3}DC.$

The distance of G from $BC = NC = AC - AN = \frac{1}{3}AC$

$$= \frac{2}{3}$$

and the distance of G from $AC = GN = \frac{2}{3}DC = \frac{1}{3}BC$

$$= \frac{1}{3}.$$

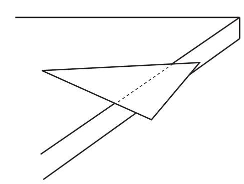
The distances of G from BC and AC are $\frac{2}{3}$ m

and $\frac{1}{3}$ m respectively.

Activity 11   Finding the centre of gravity of a triangle

You need cardboard, scissors, thread and a table.

Cut a triangle out of cardboard.

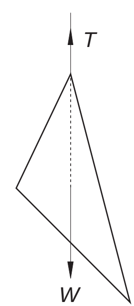


Place the triangle so as to partly overhang the edge of the table.

Adjust it so that it is just on the point of toppling and draw a line on it to indicate the edge of the table.

Why is the centre of gravity of the triangle on this line?

Orientate the triangle another way and so find a second line on which the centre of gravity lies.



The intersection of the two lines gives the centre of gravity of the triangle.

Repeat with the triangle orientated a third way and check that the third line also passes through G.

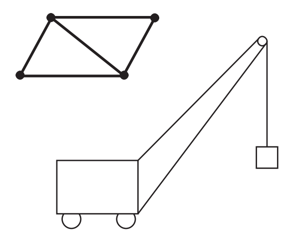
Draw the medians of your triangle and see if G is at their point of intersection.

As a further check, suspend the triangle from different points by a thread. Since the triangle is in equilibrium, the tension, T, in the thread and the weight, W, of the triangle must act along the same line, so the centre of gravity lies on the vertical line through the point of suspension.

This technique is used by biologists to determine the centres of gravity of insects.

9.12 The centre of gravity of composite bodies

A composite body is one made up of two or more simple bodies joined together, for example a framework or a crane. The weight of the body is the sum of the weights of the bodies that make it up.

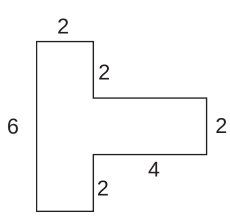


For the purposes of calculating the centre of gravity of the composite body each of the simpler bodies can be regarded as equivalent to a particle of the same weight situated at its centre of gravity. When the composite body is made up of two bodies of weights W_1 and W_2 and centres of gravity G_1 and G_2 its centre of gravity coincides with the centre of gravity of two particles of weights W_1 and W_2 at G_1 and G_2 respectively.

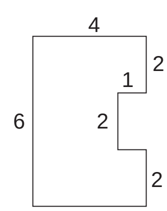
Example

Find the positions of the centres of gravity of the T- and U- shapes shown. Lengths are in metres.

(a)

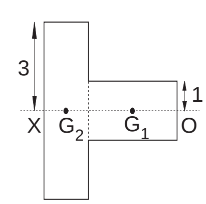


(b)



Solution

(a) Since the T-shape is symmetric about the line OX its centre of gravity lies on OX.

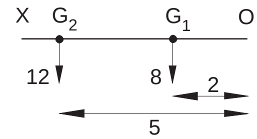


The shape can be divided into two rectangles as shown and these rectangles replaced by particles at their centres of gravity G_1 and G_2, the weights of the particles being proportional to the areas of these rectangles.

The resultant of these weights is proportional to the sum of these areas and acts at the centre of gravity, G, of the shape.

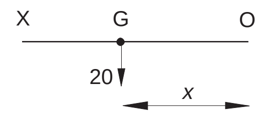
Equating moments about O for these two systems gives

$$20x = 8 \times 2 + 12 \times 5 = 76,$$



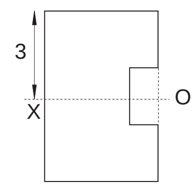
so that

$$x = \frac{76}{20} = 3.8.$$



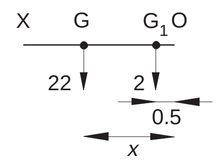
The centre of gravity of the shape is on OX, distance 3.8 m from O.

(b) The centre of gravity of the shape lies on its line of symmetry OX.



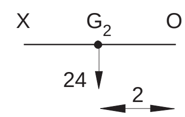
The U-shape can be regarded as a large rectangle with a small rectangle cut out as shown.

The U-shape and the small rectangle can be replaced by particles at the centres of gravity G_1 and G_2, where weights are proportional to the areas.



The resultant of these weights is proportional to the area of the large rectangle and acts at its centre of gravity G_2.

Since the areas of the large and small rectangles are 24 m^2 and 2 m^2, the area of the U-shape is 22 m^2.



Taking moments about O for the two systems gives

$$24 \times 2 = 2 \times \frac{1}{2} + 22 \times x,$$

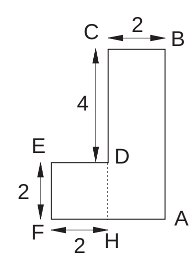
so that $\quad 22x = 48 - 1 = 47$

and $\quad x = \frac{47}{22} = 2.14.$

The centre of gravity of the shape is on OX, distance 2.14 m from O.

Example

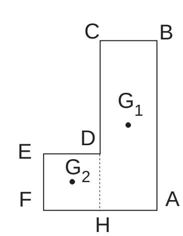
Find the distance from AB of the centre of gravity of the L-shape shown. Lengths are in metres.



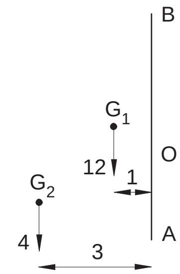
Solution

The area of the L-shape is 16 m^2.

The L-shape can be regarded as made up of two rectangles ABCH and DEFH of areas 12 m^2 and 4 m^2 respectively, with centres of gravity G_1 and G_2. The centre of gravity, G, of the shape lies on $G_1 G_2$.



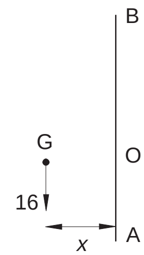
Imagine the shape to be in a vertical plane. Then the weight of the shape acts at its centre of gravity, G, and is parallel to AB. This is equivalent to the weights of the two rectangles acting at G_1 and G_2 and also parallel to AB.



The moments of the two systems about any point O on AB are equal. If x is the distance of the line of the weight of the shape from AB, then

$$16x = 12 \times 1 + 4 \times 3 = 24$$

so $$x = \frac{24}{16} = \frac{3}{2}.$$



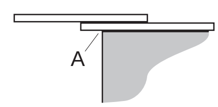
The centre of gravity of the shape is distant 1.5 m from AB.

Activity 12 Overhanging rules

You need a set of metre rules (as near equal in weight as possible) and a table. Lay one rule on the table at right angles to its edge and find out what is the most overhang you can get.

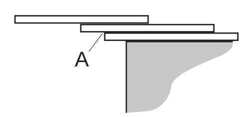
How far is the centre of gravity of the rule from the table edge when it is just about to topple?

Find out the most overhang you can get with two rules.



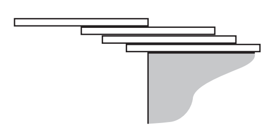
How far is the centre of gravity of the two rules from the table edge when they are just about to topple?

Imagine the two rules put on the table and then the first two rules put as they are on top of this third rule with the point A of the bottom rule, originally at the edge of the table, now at the overhanging end of the third rule. Assuming all three rules are equal in length and weight, calculate by how much the third rule overhangs the table when the rules are just on the point of toppling.



Check your result experimentally.

Now imagine the three rules lifted off as they are and then placed on top of the fourth rule on the table, again with the point of the lowest of the three rules originally at the table edge now at the end of the fourth rule. Calculate the amount the fourth rule overhangs the table when the rules are just on the point of toppling.

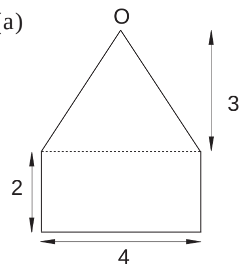


Check your result experimentally. Can you get the top rule to completely overhang the table?

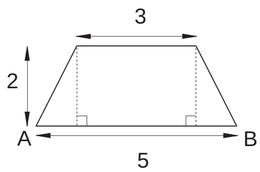
Exercise 9H

1. Find the centres of gravity of the shapes shown. Lengths are in metres.

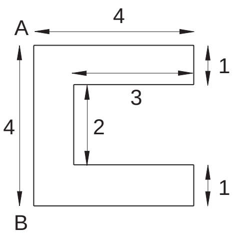
(a)



(b)

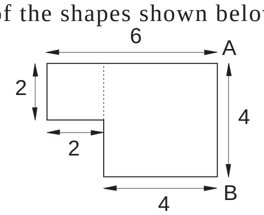


(c)

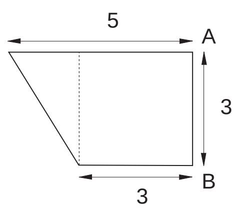


2. Find the distances from AB of the centres of gravity of the shapes shown below.

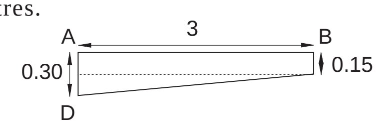
(a)



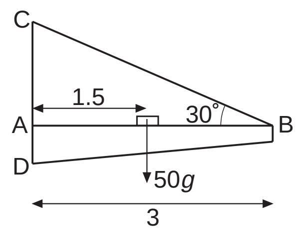
(b)



3. The jib of a wall crane has the shape and dimensions shown in the diagram, lengths being in metres.



It is made of 20 mm thick steel plate of density 800 kg m^{-3}. Find its mass and the distance of its centre of gravity from AD.

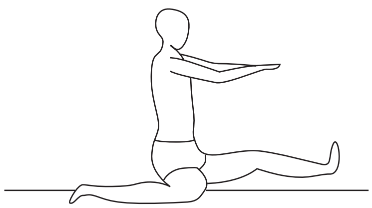


The jib is hinged to the wall at A by a pin-joint. A trolley of weight 50g N is on the jib 1.5 m from A. Find the tension in the rod BC and the horizontal and vertical components of the contact force of the wall in the joint at A. (This model of the wall crane takes account of the weight of the jib.)

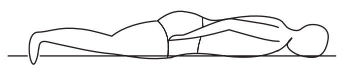
9.13 Some applications of centres of gravity

High jumping

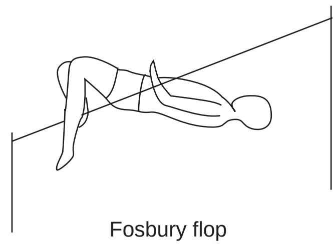
The diagrams below show three high jump techniques - the scissors, the straddle and the Fosbury flop.



Scissors



Straddle



Fosbury flop

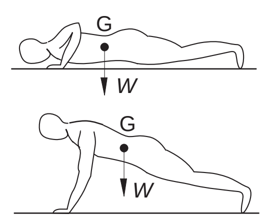
Where are the centres of gravity of the jumpers likely to be, relevant to the bars?

Which is likely to be the most and which the least effective technique?

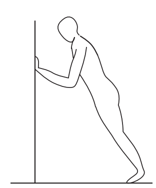
If you have access to a slow motion video of high jumping, you might like to look at it.

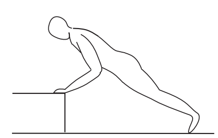
Keeping fit

Exercises help you to keep fit and improve your performance at sport. Several exercises involve raising the whole or part of your body. During a press-up from the floor the body rotates about the feet due to the action of the arms. Muscles in the arms must overcome the moment about the feet of the body's weight at the centre of gravity, G; the smaller this moment, the easier the exercise is in terms of the strain on the arms.

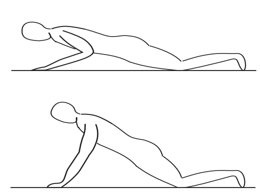


It is easier to do press-ups against a wall or a bar than against the floor.



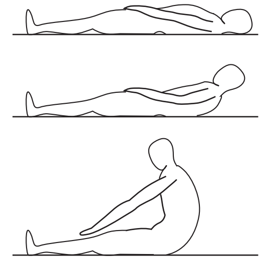


Why? Which is easier, the wall or the bar?



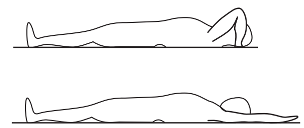
Why is it easier to do press-ups using the knees rather than the feet?

Activity 13   The trunk curl - sit up



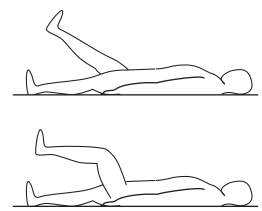
Try the trunk curl - sit up exercise shown in the three diagrams opposite.

The first stage of this exercise involves a trunk curl as shown in the second diagram, before the sit up is completed. What is the advantage of this?



Now try a trunk curl - sit up with your arms in the positions shown.

Why are these more difficult to do? Which is most difficult?



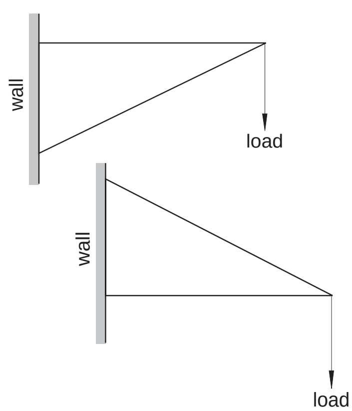
Now try an easier exercise, just raising one leg at a time.

Why is this easy?

Why is it even easier when the leg is bent?

Climbing trees

When a squirrel climbs a tree, it needs to overcome gravity. Its weight is balanced by an equal and upward force on its hind feet which it sets up either by digging in its claws or by friction. Since the squirrel's centre of gravity is not directly above its claws, this upward force and the weight exert a clockwise moment on the squirrel. To balance this, the squirrel must pull on the tree trunk with its forefeet and push with its hind feet.

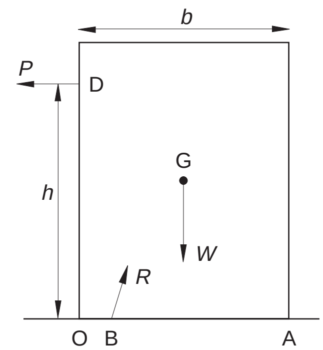


How does a woodpecker stay in equilibrium on a tree? Why does it rest its tail on the tree trunk?

Compare the contact forces exerted on the wall by the two structures on the right with those exerted by the squirrel and the woodpecker. Which best models the squirrel and the woodpecker?

9.14 Sliding and toppling of a block

In Activity 10 you found experimentally when a block slides and when it topples. To obtain mathematical criteria for sliding and toppling it is necessary to know the forces acting on the block.

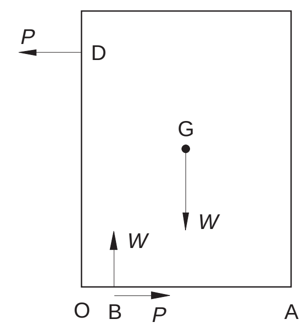


These are

the weight, W, acting vertically downwards through the centre of gravity, G, of the block;

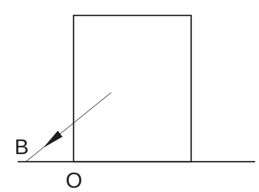
the longitudinal pull, P, on the block acting at the point D;

the force, R, the ground exerts on the block acting at a point B in the base OA of the block.

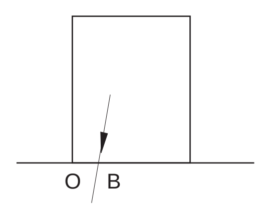


When the block is in equilibrium, the resultant of W and P must be equal and opposite to R and act along a line passing through B. The force due to the ground therefore has a normal component $N = W$ upwards to balance the weight and a friction component $F = P$ horizontally to balance the pull, this component being due to friction between the ground and the block.

For the block not to topple, B must lie between O and A. Otherwise, the resultant of W and P has an anticlockwise moment about O and the block topples about O.



topples



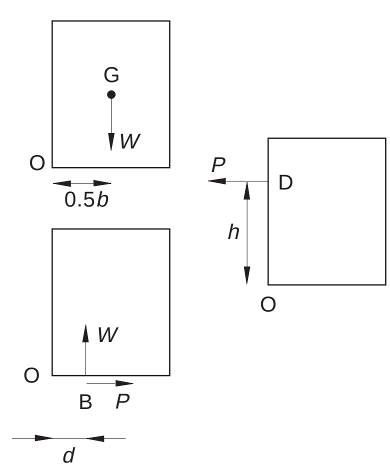
does not topple

The position of B is found by taking moments about O:

the moment of the weight W about O is clockwise and is $\frac{-Wb}{2}$, b the width of the block;

the moment of the pull P about O is anticlockwise and is Ph;

the moment of the force due to the ground about O is anticlockwise and is Wd, where $\text{OB} = d$.



Provided the block is in equilibrium, the sum of these moments is zero. Hence

$$\frac{-Wb}{2} + Ph + Wd = 0,$$

which gives

$$Wd = \frac{Wb}{2} - Ph$$

or $$d = \frac{b}{2} - \frac{Ph}{W}.$$

Since B is between O and A, $\text{OB} = d \geq 0$, so

$$\frac{b}{2} \geq \frac{Ph}{W}$$

or $$\frac{Wb}{2} \geq Ph.$$

This says that the block does not topple provided the moment of the weight about O is greater than or equal to the moment of the pull about O.

The block is just on the point of toppling when $d = 0$ or $\frac{Wb}{2} = Ph$. This gives the value of P for toppling as

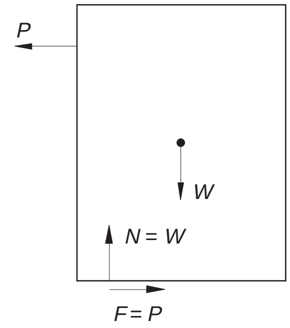
$$P = \frac{Wb}{2h},$$

in other words, P is inversely proportional to h.

Is this the relation you found in Activity 10?

When does the block slide?

The block slides when there is not enough friction to hold it in position.



The force on the block due to the ground has a vertical component $N = W$, the normal contact force, and a horizontal component $F = P$, due to friction.

The law of friction says that

$$F \le \mu N,$$

where μ is the coefficient of friction. If the block has not already toppled, it is on the point of sliding when the friction becomes limiting, that is, $F = \mu N$.

It does not slide, however, provided $F \le \mu N$, that is,

$$P \le \mu W$$

or $$\frac{P}{W} \le \mu.$$

It does not topple provided

$$\frac{P}{W} \le \frac{b}{2h}.$$

If $\frac{b}{2h} > \mu$, then, as P increases from zero, $\frac{P}{W}$ reaches the value μ before the value $\frac{b}{2h}$. The block slides. Similarly, if $\frac{b}{2h} < \mu$, $\frac{P}{W}$ reaches the value $\frac{b}{2h}$ before the value μ. The block topples.

> As P increases from zero, the block slides first if $\frac{b}{2h} > \mu$ and topples first if $\frac{b}{2h} < \mu$.

Activity 14 Sliding and toppling of a block

You need your data from Activity 10.

Choose values of P and h for which the block slides before it topples.

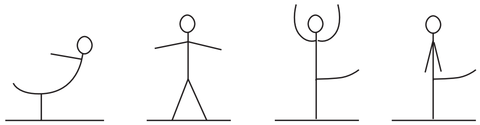
Use these values to find the coefficient of friction between the block and the table.

Investigate how well the remaining data you obtained in Activity 10 fit the criteria for sliding and toppling of a block.

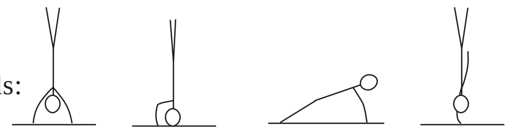
Activity 15 Gymnasts

Arrange each group of gymnasts in the order in which they are most in danger of toppling over, explaining why you choose this order.

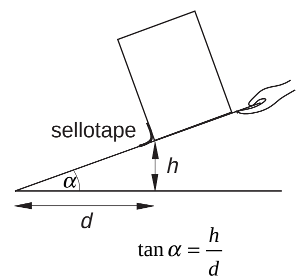
Balancing on the feet:



Balancing on the hands:



Activity 16 Toppling packets



You need two pieces of wood or rules, Sellotape, rectangular packets of soap flakes, cereal, sugar, jelly, etc. (alternatively you can use multilink).

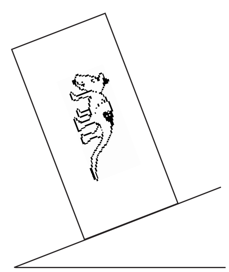
Sellotape one piece of wood to the other (or to the table) so as to make an inclined plane when you hold it.

Investigate practically the angles α at which packets of different heights topple.

Use a piece of Sellotape at the front of the packet to prevent it sliding.

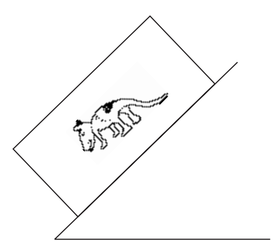
The angle α can be found by measuring h and d as in the diagram.

Obtain a theoretical criterion for the toppling of a rectangular packet on an inclined plane.



Check how well your theoretical criterion agrees with your actual results. Use your criterion to predict the angle at which a packet should topple and see what actually happens.

Place a packet with its largest side at right angles to the plane and then along the plane. In each case measure the angle at which the packet topples.



Can you find a relation between these angles?

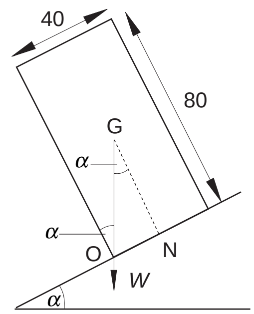
Can you confirm this relation theoretically?

Example

A uniform rectangular block of width 40 mm and height 80 mm is placed on a plane which is then gently raised until the block topples. What angle does the plane make with the horizontal when this occurs?

Solution

The block is about to topple when the moment about its lowest point, O, is zero. This is when the line along which the weight acts, the vertical through G, passes through O.



In triangle ONG the side $ON = 20,\ NG = 40$

and so

$$\tan\alpha = \frac{ON}{NG} = 0.5,$$

which gives $\alpha = 26.6°$. From the diagram the angle the plane makes with the horizontal is also $\alpha = 26.6°$.

Exercise 9I

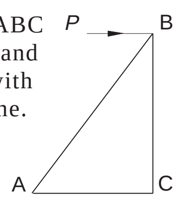
1. A block of mass 1 kg, width 40 cm and height 100 cm, rests on a table. Find the horizontal force, P, which causes it to topple when:

   (a) P acts at the top of the block;

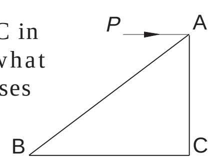
   (b) P acts halfway up.

   What is the least value of the coefficient of friction for which the block topples rather than slides whichever of the forces (a) or (b) is applied?

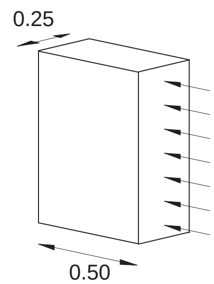
2. A right-angled triangular sheet ABC has sides $AC = 30$ cm, $BC = 40$ cm and mass 1 kg. It is placed vertically with the side AC along a horizontal plane. What horizontal force P at B causes the sheet just to topple?



   When the sheet has the side BC in contact with the plane, what horizontal force P at A then causes the sheet to topple?



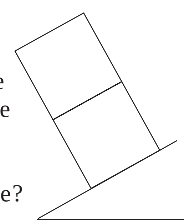
3. A brick column is 0.5 m wide and 0.25 m deep and weighs 18 000 N per cubic metre. There is a uniform wind pressure of 750 N per square metre on one side as shown in the diagram opposite. The column rests on a block but is not attached to it.



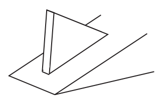
   What is the greatest height the column can be if it is not to topple?

   Assume the wind pressure produces a horizontal force on the column whose magnitude is the wind pressure times the area over which it acts, the force acting at the point of intersection of the diagonals of the force.

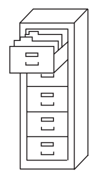
4. A tower is made of multilink cubes, each of side 20 mm, and is placed on a plane. The plane is gently raised until the tower topples. At what angle of the plane to the horizontal does a tower of 2 cubes topple? At what angles do towers of 3 and 10 cubes topple?



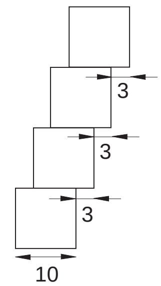
5. An equilateral triangular sheet is placed vertically with one side in contact with a horizontal plane. If the plane is raised gently, at what angle to the horizontal does the sheet topple?



6. A filing cabinet has four drawers, each of which has mass W kg when empty, the mass of the rest of the cabinet being $3W$ kg. When the bottom three drawers are empty and the papers in the top drawer weigh six times as much as the drawer itself, how far can this drawer be opened without the cabinet toppling? Treat the drawers as rectangles.

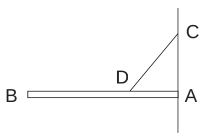


7. A pile of equal cubical blocks, each of edge 10 cm, is made by placing the blocks one on top of the other, with each displaced a distance 3 cm relative to the block below.
Show that a pile of four blocks does not topple but one of five does.



9.15 Miscellaneous Exercises

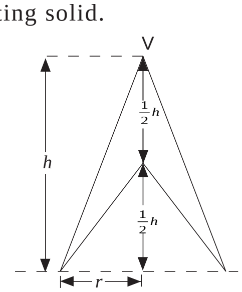
1. A uniform shelf of length 46 cm is hinged to a vertical wall as shown. The shelf is supported by a rod and the tension in the rod is 100 N.
If $AD = 15$ cm and $A\hat{D}C = 50°$, find the mass of the shelf.



2. A circular hole of radius 2 m is made in a circular disc of radius 8 m. If the centre of the hole is 4 cm from the centre of the disc, find the position of the centre of gravity of the disc with its hole.

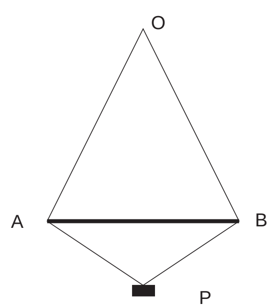
3. A hollow cylinder of diameter 5 cm is placed on a rough plane which is inclined at $30°$ to the horizontal. What is the maximum height of the cylinder if it is not to topple over?

4. The diagram shows a solid uniform right circular cone of height h, base radius r and vertex V from which has been removed a solid coaxial cone of height $\frac{1}{2}h$, base radius r.
Find the distance from V of the centre of mass of the resulting solid.



(AEB)

5. [In this question you should assume $g = 9.8\text{ms}^{-2}$.]



The diagram shows a body, P, of mass 13 kg, which is attached to a continuous inextensible string of length 15.4 m. The string passes over a small smooth peg O and the hanging portions of the string are separated by a heavy uniform horizontal rigid rod AB, which is 4 m long, the string fitting into small grooves at the ends of the rod. Given that in the equilibrium position $AP = BP = 2.5$ m, show that

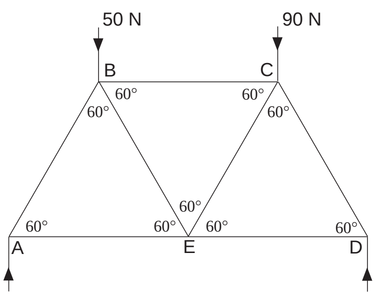
(a) $\sin P\hat{A}B = \frac{3}{5}$ and $\sin O\hat{A}B = \frac{12}{13}$;

(b) the tension in the string has magnitude $\frac{637}{6}$ N.

(c) Hence find the mass of the bar AB.

(AEB)

6. The diagram shows a framework consisting of seven equal smoothly jointed light rods AB, BC, DC, DE, AE, EB and EC. The framework is in a vertical plane with AE, ED and BC horizontal and is simply supported at A and D. It carries vertical loads of 50 N and 90 N at B and C respectively.



Find

(a) the reactions at A and D;

(b) the magnitudes of the forces in AB, AE and BC.

(AEB)

7. A uniform solid right circular cylinder of radius a and height $3a$ is fixed with its axis vertical, and has a uniform solid sphere of radius r attached to its upper face. The sphere and the cylinder have the same density and the centre of the sphere is vertically above the centre of the cylinder. Given that the centre of mass of the resulting composite body is at the point of contact of the sphere with the cylinder, find r in terms of a.

(AEB)

8. An equilateral triangular frame ABC is made up of three light rods, AB, BC and CA which are smoothly jointed together at their end points. The frame rests on smooth supports at A and B, with AB horizontal and with C vertically above the rod AB. A load of 50 N is suspended from C. Find the reactions at the supports and the tensions or thrusts in the rods.

(AEB)

9. A composite body B is formed by joining, at the rims of their circular bases, a uniform solid right circular cylinder of radius a and height $2a$ and a uniform right circular cone of radius a and height $2a$. Given that the cylinder and the cone have masses M and λM respectively, find

(a) the distance of the centre of mass of B from the common plane face when $\lambda = 1$;

(b) the value of λ such that the centre of mass of B lies in the common plane face.

(AEB)

10. The base of a uniform solid hemisphere has radius $2a$ and its centre is at O. A uniform solid S is formed by removing, from the hemisphere, a solid hemisphere of radius a and centre O. Determine the position of the centre of mass of S. (The relevant result for a solid hemisphere may be assumed without proof.) (AEB)

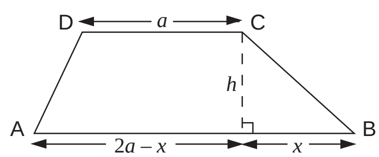
11. A light square lamina ABCD of side $2a$ is on a smooth horizontal table and is free to turn about a vertical axis through its centre O. Forces of magnitude P, $2P$, $3P$ and $7P$ act along the sides $\vec{AB}$, $\vec{BC}$, $\vec{CD}$ and $\vec{AD}$ respectively. Find the magnitude of the couple required to maintain equilibrium and also the magnitude of the reaction at O.

(AEB)

12. A uniform rod AB of weight 40 N and length 1.2 m rests horizontally in equilibrium on two smooth pegs P and Q, where $AP = 0.2$ m and $BQ = 0.4$ m. Find the reactions at P and Q. Find also the magnitude of the greatest vertical load that can be applied at B without disturbing the equilibrium.

(AEB)

13. The diagram shows a uniform plane trapezium ABCD, in which AB and DC are parallel and of lengths $2a$ and a respectively. The foot of the perpendicular from C onto AB is E, $CE = h$ and $BE = x$.



Prove that the distance of the centre of mass of the trapezium from AB is $\frac{4}{9}h$. (AEB)

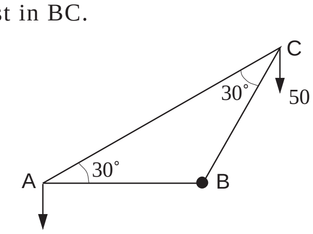
14. ABCD is a square of side 2 m. Forces of magnitude 2 N, 1 N, 3 N, 4 N and $2\sqrt{2}$ N act along $\vec{AB}$, $\vec{BC}$, $\vec{CD}$, $\vec{DA}$ and $\vec{BD}$ respectively. In order to maintain equilibrium a force F, whose line of action cuts AD produced at E, has to be applied. Find

(a) the magnitude of F;

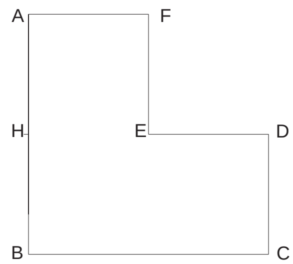
(b) the angle F makes with AD;

(c) the length AE. (AEB)

15. The diagram shows three light rods which are smoothly jointed together to form a triangular framework ABC in which the angles BAC and BCA are both 30°. The framework can turn in a vertical plane about a horizontal axis through B. When a load of 50 N is suspended from C the framework is kept in equilibrium with AB horizontal by means of a vertical force of magnitude P N applied at A. Determine P and the thrust in BC.



16. The diagram shows a uniform L-shaped lamina ABCDEF, of mass $3M$, where $AB = BC = 2a$ and $AF = FE = ED = DC = AH = a$.



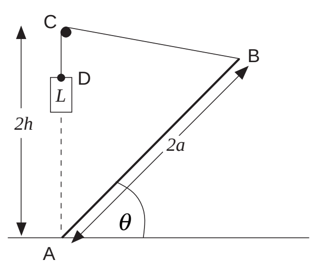
Find the perpendicular distances of the centre of mass of the lamina from the sides AB and BC. The lamina is suspended freely from H, the mid-point of AB, and hangs in equilibrium.

(a) Show that the tangent of the angle which the side AB makes with the horizontal is $\frac{1}{5}$.

(b) When a particle of mass m is attached to F the lamina hangs in equilibrium with AB horizontal. Find m in tems of M.

(c) The particle is now removed and the lamina hangs in equilibrium with BE horizontal when a vertical force of magnitude P is applied at B. Find P in terms of M and g.

17. The diagram shows a uniform rod AB of weight W and length $2a$ with the end A resting on a rough horizontal plane such that AB is inclined at an angle θ to the horizontal. The rod is maintained in equilibrium in a vertical plane in this position by means of a light inextensible string BD which passes over a small peg at C and which carries a load L at D. The peg is at height $2h$ vertically above A.



(a) Find, by taking moments about C and B, the horizontal and vertical components of the reaction on the rod at A in terms of W, a, h and θ. Given that the coefficient of friction at A is μ, show that

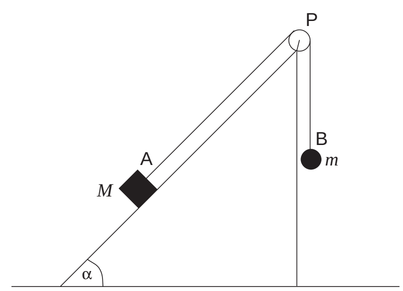
$$\frac{a\cos\theta}{h + a\sin\theta} \le \mu.$$

Hence find, in terms of h and a, the least value of μ so that equilibrium is possible for any angle θ satisfying $0 \le \theta \le 60°$.

(b) For the case when $a = \frac{1}{2}h$ and angle ABC is $90°$, find in terms of W, the load L and the vertical component of the force acting on the peg C.

(AEB)

18. The diagram shows a fixed rough inclined plane whose lines of greatest slope make an angle α with the horizontal. The plane has a smooth pulley P fixed at its highest point. Two particles, A and B, of mass M and m respectively, are attached one to each end of a light inextensible string which passes over the pulley. The masses are at rest with A in contact with the plane, B hanging freely vertically below P and the portion of string AP parallel to a line of greatest slope of the plane.



Find, in terms of M, m, g and α, the components, normal and parallel to the plane, of the force exerted on A by the plane. Deduce that

$$\sin\alpha - \mu\cos\alpha \le \frac{m}{M} \le \sin\alpha + \mu\cos\alpha,$$

where μ is the coefficient of friction.

It is found that when $\alpha = 60°$, A is on the point of moving down the plane but when $\alpha = 30°$, A is on the point of moving up the plane. Find μ.

(AEB)

ANSWERS

The answers to the questions set in the Exercises are given below. Answers to questions set in some of the Activities are also given where appropriate.

1 MODELLING AND MECHANICS

Exercise 1A

1. (a) All four forces are equal in magnitude.
   (b) They all act in the same line, forming two pairs opposite in direction.

Exercise 1B

1. 2.8×10^{-9} N
2. 33.2 N, which is 4.4% of his weight on the Earth's surface.
3. 1.64 *m* newtons.
4. 6.0 to 1
5. $\frac{9}{10}$ of the distance; that is 3.6×10^{5} km.

Exercise 1C

1. (a) and (b) are consistent.
2. (a) ML^2T^{-2} (b) ML^2T^{-2}
   (c) MLT^{-1} (d) $ML^{-1}T^{-2}$
3. $x=-1,\ y=\frac{1}{2},\ z=-\frac{1}{2}$

2 ONE-DIMENSIONAL MOTION

Exercise 2A

2. (a) 0 (b) 2 ms^{-2} (c) $-\frac{5}{3}$ ms^{-2} (d) 437.5 m
3. (c) –6 ms^{-2} (d) 148 m
4. Approx. 600 m
5. 60 m, if the pedestrian is to be able to cross at 3 mph before the arrival of a vehicle.
6. (a) $v(1)=1$ (speed of 1 ms^{-1} in a forward direction)
   $v(2)=-3$ (speed of 3 ms^{-1} in a reverse direction)
   (b) When $t=\frac{5}{4}$ (after $\frac{5}{4}s$)
   (c) $-4\ ms^{-2}$ all times.
7. (a) $v(1)=-2,\quad v(2)=0$
   $t=2$
   2 at all times
   (b) $v(1)=6,\quad v(2)=0$
   $t=2$ and $t=3$
   $-9\ ms^{-2},\quad -3\ ms^{-2}$
8. (a) $v=13-10t,\quad s=-6+13t-5t^2$
   (b) $s(2)=0,\quad v(2)=-7$

Exercise 2B

1. 0.417 ms^{-2}
2. 361 m
3. (a) 890 m (2 sf)
   (b) 67 s (2 sf)
4. 1200 m
5. (a) 2.92 ms^{-2}, 18.7 ms^{-1}
   (b) 8.55 s
6. (a) 16.3 ms^{-1}
   (b) 2.21 ms^{-2}
7. 0.580 ms^{-2}, 15.3 ms^{-1}
8. Assume that the record holder accelerated for 60 m at uniform rate.
   (a) 17.14 ms^{-1}, (b) 2.45 ms^{-2} or 1.64 ms^{-2}, depending on method. The answer to (b) shows that the model must be modified. Assume that the record holder accelerates to maximum speed before 60 m. Then the answer becomes 3.24 ms^{-2} (over 3.53 s and 20.2 m).

9. 9.3 s

10. $-10.7\ \text{ms}^{-2}$, 1.68 s

Exercise 2C

1. (a) 2 s (b) a further 2 s
2. 2.08 s, $24.8\ \text{ms}^{-1}$
3. 6.53 s, $-40.3\ \text{ms}^{-1}$
5. 667 m
6. (a) $4.01 \times 10^{14} \left(\text{m}^3\ \text{s}^{-2}\right)$
   (b) 32 km
7. $\frac{5}{3}h$
8. (a) The one thrown downwards.
   (b) Both have the same speed.
   (c) 5.4 m, 5 m

Exercise 2D

1. (a) $12.5\ \text{ms}^{-2}$ (b) 0 (c) 469 m
2. 350 - 450 N
3. (a) $0.494\ \text{ms}^{-2}$ (b) 9.88×10^4 N
5. (a) $0.0675\ \text{ms}^{-2}$ (b) $0.54\ \text{ms}^{-1}$, 10.8 m
6. (a) $30\ \text{ms}^{-2}$ (b) $150\ \text{ms}^{-2}$ (c) 0
7. 1400 N
8. 4890 N.

Exercise 2E

1. $4\ \text{ms}^{-2}$
2. An increase of weight by 20%, then normal weight; in the final stage you leave the floor with an acceleration upwards relative to the lift of $1\ \text{ms}^{-2}$.
3. 445 N
4. The box slides towards the front of the trolley with an acceleration relative to the trolley of $0.75\ \text{ms}^{-2}$. It falls off after 3.27 s.
5. (a) $0.0833\ \text{ms}^{-2}$ (b) 333 N
6. (a) $2.5\ \text{ms}^{-2}$ (b) 22.5 N (c) $-2\ \text{ms}^{-2}$
7. (a) 1100 N (b) 400 N

Exercise 2F

2. (a) $y = t^3 + C$ (b) $y = \sin t + C$
   (c) $y = e^t + C$ (d) $y = \frac{1}{t} + C$

Exercise 2G

1. (a) $y = \frac{t^4}{2} + 3t + C$ (b) $y = -\frac{1}{2}\cos 2t + C$
   (c) $y = \frac{2}{3}(1+3t)^{\frac{1}{2}} + C$ (d) $y = \frac{(1+2t)^{\frac{3}{2}}}{3} + C$
   (e) $y = \ln(1+t) + C$ (f) $y = \frac{1}{4}\ln\left(1+2t^2\right) + C$
2. (a) $y = \frac{3t^2}{2} + 4t$ (b) $y = \frac{t^5}{5} - \frac{1}{4t^3} + 1.05$
   (c) $y = \frac{1}{2e^{2t}} + 2$ (d) $y = -\cos 2t + 1$

Exercise 2H

1. (a) $0.75\ \text{kg ms}^{-1}$ (b) $4\ \text{ms}^{-1}$
2. $2\ \text{ms}^{-1}$
3. $4.0\ \text{kg ms}^{-1}$
4. $0.714\ \text{kg ms}^{-1}$ upwards
5. $490\ \text{kg ms}^{-1}$
6. $0.618\ \text{ms}^{-1}$
7. $2.14\ \text{ms}^{-1}$

Miscellaneous Exercises

1. (a) $13.0° < \theta < 16.7°$, where θ is the angle made with BJ.
   (b) $\theta < 16.7$, $\theta \neq 0$
   (c) If reaction time is ignored,
   $\sin^{-1}\left(\frac{v_J}{v_B}\right) < \theta < 16.7°$.
2. (a) 25 m (b) 20 s (c) $50\ \text{ms}^{-1}$, $62.5\ \text{ms}^{-1}$
3. 15.2 m from the front of the box.
4. (a) 44.8 m (b) 40 m
5. (a) Assuming acceleration and deceleration are each $2\ \text{ms}^{-2}$, D=87 (2 sf).
6. 32.5
7. (a) $1.2\ \text{ms}^{-2}$ (b) 970 m
8. 5
9. (a) 1.25×10^5 N, $312.5\left(\text{Nm}^{-2}\ \text{s}^2\right)$
   (b) $0.456\ \text{ms}^{-2}$
10. $20\ \text{ms}^{-1}$, 2.77 s
11. $4.47\ \text{ms}^{-1}$, 1.39 m
12. 3140 N

13. 9.44 ms^{-2}

14. (b) 2.2 ms^{-1}

15. 0.143 ms^{-2}, 1290 N, 2570 N

16. 2 ms^{-2}, 16 N, 2.83 ms^{-1}

17. More than ten trucks.

18. (a) 6.84 ms^{-2} (b) 81.7 ms^{-2}

19. The case accelerates forward on the trolley at 7 ms^{-2} for 5 s, or until it falls off.

21. (a) 48.9 kph (b) 2.31 ms^{-1}, 4.15×10^5 N

22. (a) 1 ms^{-1} (b) 8.6 cm

23. 0.4Ns: 6.3 ms^{-1}

24. $2m\sqrt{\left(\frac{2}{3}gh\right)}$

25. $t_{acc} = 4v$; $t_{dec} = v$; $v^2 - 60v + 800 = 0$; $v = 20$ ms^{-1}

26. 0.25 ms^{-2}; 0; -0.5 ms^{-1}; 8000 m

27. (a) 1200 km s^{-2} (b) 3 mins (c) 0.5 km (d) 5 mins

28. $v = \left(10x^2 - x^4 - 9\right)^{\frac{1}{2}}$; 1, 3

29. (a) $\frac{3u}{64}$ ms^{-2}, $\frac{u}{32}$ ms^{-2} (c) 3

30. 1.2 ms^{-2}; -1.8 ms^{-2}; 10 s; 1305 m

31. 9.2 km; 440 s

3 VECTORS 1

Exercise 3A

1. (a) 13 miles, 067.4° (b) $\begin{pmatrix}12\\5\end{pmatrix}$

2. (a) 6.71 miles, 333.4° (b) $\begin{pmatrix}-3\\6\end{pmatrix}$

3. (a) 6.57 miles, 122.6° (b) $\begin{pmatrix}5.54\\-3.54\end{pmatrix}$

Exercise 3B

1. (a) $4\mathbf{i} + 2\mathbf{j}$ (b) $\mathbf{i} + 3\mathbf{j}$ (c) $-3\mathbf{i} + \mathbf{j}$ (d) $3\mathbf{i} - \mathbf{j}$
(e) $-3\mathbf{i} - 5\mathbf{j}$ (f) $4\mathbf{i} + \mathbf{j}$ (g) $\mathbf{i} - 4\mathbf{j}$ (h) $-\mathbf{i} + 4\mathbf{j}$
(i) $2\mathrm{i} - 3\mathrm{j}$ (j) $7\mathbf{i}$ (k) $\frac{1}{2}\mathbf{i} - 2\mathbf{j}$
(l) $30\mathbf{i} + 20\mathbf{j}$

2. (a) $\overrightarrow{QO} = -3\mathbf{i} - \mathbf{j}$, $\overrightarrow{QC} = -5\mathbf{i} - 3\mathbf{j}$, $\overrightarrow{DQ} = \mathbf{i} + 2\mathbf{j}$
(b) $\overrightarrow{RO} = -p\mathbf{i} - q\mathbf{j}$, $\overrightarrow{RC} = -(p+2)\mathbf{i} - (q+2)\mathbf{j}$
$\overrightarrow{AR} = (p-4)\mathbf{i} + (q-2)\mathbf{j}$

4. (a) $\overrightarrow{LD}$, $\overrightarrow{DM}$, $\overrightarrow{IJ}$, $\overrightarrow{GE}$, $\overrightarrow{EF}$, $\overrightarrow{FC}$
(b) $\overrightarrow{MJ}$, $\overrightarrow{PD}$, $\overrightarrow{DI}$, $\overrightarrow{BN}$, $\overrightarrow{NK}$, $\overrightarrow{HG}$
(c) $\overrightarrow{AP}$, $\overrightarrow{CI}$, $\overrightarrow{KF}$ (d) $\overrightarrow{FJ}$, $\overrightarrow{HL}$ (e) $\overrightarrow{CD}$, $\overrightarrow{NO}$
(f) $\overrightarrow{BM}$, $\overrightarrow{NJ}$

Exercise 3C

1. $3\sqrt{10}$, $-71.6°$; $\sqrt{5}$, $-153.4°$;
$2\sqrt{3}$, 60°; $\sqrt{41}$, 141.3°

2. $\sqrt{13}$, $\sqrt{13}$, $\sqrt{2}$, $\sqrt{a^2 - 4a + 53}$; 1 or 3

3. (a) $-2\mathbf{r}$ (b) $\mathbf{p} - \mathbf{q} + \mathbf{r}$ (c) $\mathbf{p} - \mathbf{q} - \mathbf{r}$
(d) $3\mathbf{p} - 4\mathbf{q}$ (e) $-\mathbf{p} + 2\mathbf{q}$ (f) $\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q} + \mathbf{r}$
(g) $\frac{1}{2}\mathbf{p} - \frac{5}{2}\mathbf{q} + \mathbf{r}$ (h) $-3\mathbf{p} + 4\mathbf{q}$ (i) $\mathbf{p} - \mathbf{q} - \mathbf{r}$
(j) $-\frac{1}{2}\mathbf{p} - \frac{1}{2}\mathbf{q} + \mathbf{r}$

Exercise 3D

1. (a) $5\mathbf{i} - 3\mathbf{k}$ (b) $5\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$ (c) $10\mathbf{i} + 2\mathbf{j} - 12\mathbf{k}$

2. (a) $5\mathbf{i} - 3\mathbf{j} - 8\mathbf{k}$ (b) $-6\mathbf{i} - \mathbf{j} + 7\mathbf{k}$ (c) $3\sqrt{2}$
(d) $\frac{\sqrt{2}}{14}(-5\mathbf{i} + 3\mathbf{j} + 8\mathbf{k})$

3. $m = 0$, $n = -\frac{2}{3}$

4. $x = 0.8$, $y = -0.6$, $z = 1$

Exercise 3E

3. $\cos^{-1}\left(-\frac{1}{3\sqrt{3}}\right)$

4. 0

Exercise 3F

2. $-3\mathbf{i} - 3\mathbf{j}$, $-3\mathbf{i} + 3\mathbf{j}$, $3\mathbf{i} + 3\mathbf{j}$, $3\mathbf{i} - 3\mathbf{j}$, $3\mathbf{i}$

Exercise 3G

1. $24\mathbf{i}+(7-10t)\mathbf{j}$, $25\ \text{ms}^{-1}$
2. $200\ \text{ms}^{-1}$, $036.9°$
3. $61.2\ \text{ms}^{-1}$, $60\mathbf{i}-12\mathbf{j}-\mathbf{k}$
4. $100\mathbf{i}-20\mathbf{j}-30\mathbf{k}$, $107\ \text{ms}^{-1}$

Exercise 3H

1. $12\mathbf{i}+16\mathbf{j}$. The acceleration is $10\,\text{ms}^{-2}$ downwards.
2. In each case acceleration is $10\ \text{ms}^{-2}$ downwards (due to gravity).
3. $3\mathbf{i}-4\mathbf{j}$

Miscellaneous Exercises

1. (a) (i) $2\sqrt{2}$, $-45°$ (ii) 2, $60°$ (iii) $\sqrt{10}$, $108.4°$

   (b) (i) $\sqrt{3}$ (ii) $\sqrt{29}$ (iii) $\sqrt{11}$
2. (a) (i) $\frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{j})$ (ii) $\frac{1}{2}(\mathbf{i}+\sqrt{3}\mathbf{j})$ (iii) $\frac{\sqrt{10}}{10}(-\mathbf{i}+3\mathbf{j})$

   (b) (i) $\frac{\sqrt{3}}{3}(\mathbf{i}+\mathbf{j}+\mathbf{k})$ (ii) $\frac{\sqrt{29}}{29}(2\mathbf{i}-3\mathbf{j}+4\mathbf{k})$

   (iii) $\frac{\sqrt{11}}{11}(3\mathbf{i}-\mathbf{j}-\mathbf{k})$
3. (a) $-\mathbf{i}+7\mathbf{j}$ (b) $\sqrt{13}$ (c) $\frac{\sqrt{29}}{29}(2\mathbf{i}+5\mathbf{j})$
4. $a=\frac{11}{2}$, $b=-\frac{1}{2}$, $c=3$; $\frac{\sqrt{21}}{21}(\mathbf{i}+4\mathbf{j}+2\mathbf{k})$
5. (a) $4\mathbf{i}+2(\sqrt{3}-1)\mathbf{j}$ (b) $-5\mathbf{i}+3\mathbf{j}-3\mathbf{k}$

   (c) $-3\mathbf{i}-3\mathbf{j}+3\mathbf{k}$ (d) $2\mathbf{i}-\mathbf{j}-4\mathbf{k}$
6. (a) Upwards $(90°)$ (b) $-3\mathbf{i}+7\mathbf{j}$, $-\mathbf{i}+4\mathbf{j}$

   (c) $-2\mathbf{i}+5\mathbf{j}+3\mathbf{k}$
7. (a) $\sqrt{20}$, $4\mathbf{i}+2\mathbf{k}$

   (b) $3\mathbf{i}+4\mathbf{j}+10\mathbf{k}$, $-3\mathbf{i}-\mathbf{j}-5\mathbf{k}$ (c) $-4\mathbf{i}-\mathbf{j}-7\mathbf{k}$
8. (a) 26.9 (b) $\sqrt{17}$, $-76.0°$ (c) $\left(\frac{7}{2},\ -1\right)$
9. $6\mathbf{i}+2t\mathbf{k}$, 20.1
10. $\mathbf{i}-\frac{1}{t^2}\mathbf{j}$, $\frac{\sqrt{17}}{4}$
11. (a) $4t^2\mathbf{i}+8t\mathbf{j}$ (m) (b) $8t\mathbf{i}+8\mathbf{j}$ (ms^{-1})

    (c) $\mathbf{a}=8\mathbf{i}\ (\text{ms}^{-2})$
12. $\mathbf{v}=25\mathbf{i}+16\mathbf{j}-15\mathbf{k}$, $\mathbf{a}=18\mathbf{i}-6\mathbf{j}-14\mathbf{k}$
13. (a) $3\ \text{ms}^{-2}$ upwards (b) (i) $4\mathbf{i}+6\mathbf{j}+6\mathbf{k}$, (ii) $9.38\ \text{ms}^{-1}$ (c) $56.3°$
14. (a) All $20\ \text{ms}^{-1}$. The speed is cyclic, with period 5π s.

    (b) $100\mathbf{i}+200\mathbf{j}$, $100\mathbf{i}+\left(200+\frac{10\pi}{3}\right)\mathbf{j}$, $\frac{10\pi}{3}$ m
15. (a) $\mathbf{v}=2t\,\mathbf{i}+\mathbf{j}+3t^2\mathbf{k}$, $\mathbf{a}=2\mathbf{i}+6t\mathbf{k}$

    (b) $\left(9t^4+4t^2+1\right)^{-\frac{1}{2}}\left(2t\,\mathbf{i}+\mathbf{j}+3t^2\mathbf{k}\right)$
16. 0, π
17. $\mathbf{i}-2\mathbf{j}$, $11\mathbf{i}+3\mathbf{j}$, $2.5\mathbf{i}-4\mathbf{j}$
18. $8\ \text{ms}^{-1}$, $4\ \text{ms}^{-1}$
19. (a) $\frac{\pi}{2}$ (b) 0, $\frac{\pi}{2}$, π
20. $(3\mathbf{i}+4\mathbf{j})$ Ns; $\mathbf{n}=\frac{4}{5}\mathbf{i}-\frac{3}{5}\mathbf{j}$, $10\ \text{ms}^{-1}$

4 VECTORS 2

Exercise 4A

1. (a) $2\mathbf{i}-15t^2\mathbf{j}$, $-30t\,\mathbf{j}$ (b) $-3\mathbf{k}$, 0

   (c) $-\frac{2}{t^3}\mathbf{i}+4\mathbf{j}+t\,\mathbf{k}$, $\frac{6}{t^4}\mathbf{i}+\mathbf{k}$
2. $\mathbf{v}=0.5\mathbf{i}-0.2t\,\mathbf{j}$, $\mathbf{r}=0.5t\,\mathbf{i}-0.1t^2\mathbf{j}$
3. $16\mathbf{j}-2\mathbf{k}$, $80.5\ \text{ms}^{-1}$
4. (a) $(5t+2)\mathbf{i}+\mathbf{j}-\frac{t^2}{2}\mathbf{k}$,

   $\left(\frac{5t^2}{2}+2t+3\right)\mathbf{i}+(t-1)\mathbf{j}+\left(7-\frac{t^3}{6}\right)\mathbf{k}$

   (b) $\left(\frac{1}{3}+\frac{8}{3}t^{\frac{3}{2}}\right)\mathbf{j}-\frac{1}{6}t^3\mathbf{k}$,

   $-\mathbf{i}+\left(\frac{5}{3}+\frac{1}{3}t+\frac{16}{15}t^{\frac{5}{2}}\right)\mathbf{j}-\left(1+\frac{1}{24}t^4\right)\mathbf{k}$

   (c) $3\mathbf{i}+(2-10t)\mathbf{j}$, $3t\,\mathbf{i}+\left(2t-5t^2\right)\mathbf{j}$
5. (a) $12.2\ \text{ms}^{-1}$, 17.9 m (b) $7.99\ \text{ms}^{-1}$, 8.59 m

Exercise 4B

1. (a) $9\sin 30°\mathbf{i} + 9\cos 30°\mathbf{j}$
   (b) $16\cos 25°\mathbf{i} - 16\sin 25°\mathbf{j}$
   (c) $\mathbf{H} = -12\sin 41°\mathbf{i} - 12\cos 41°\mathbf{j}$,
   $\mathbf{K} = 5\cos 36°\mathbf{i} + 5\sin 36°\mathbf{j}$
   (d) $\mathbf{L} = -20\sin 15°\mathbf{i} + 20\cos 15°\mathbf{j}$,
   $\mathbf{M} = -15\cos 30°\mathbf{i} - 15\sin 30°\mathbf{j}$, $\mathbf{N} = 25\mathbf{i}$
2. $\sqrt{a^2 + 144}$, 33.0
3. (a) 8.54, 20.6° (b) 3, 144.7° (c) 29, −46.4°
   (d) 7, 63° (e) 4, 121°

Exercise 4C

1. (a) 0.1 (b) 0.15
2. 90 N
3. 0.24
4. (a) $\frac{2}{3}$ (b) 28.3 N
5. (a) 0.083 (b) 30.8 N

Exercise 4D

1. If **R** is the normal contact force, **F** the force of friction, $R = W\cos 20°$, $F + W\sin 20° = P$, $F = \mu R$
2. 0.5
3. $2 - \sqrt{2} = 0.586$
4. 1.81 kg
5. (a) 115 N (b) 117 N
6. 1.86
7. $-70\sqrt{2}\,\mathbf{i} - 70\sqrt{6}\,\mathbf{k}$

Exercise 4E

1. 17.8, 30.5°; 7.29, −6.3°; 34.8, 48.3°; 14.9, 100.1°
2. (a) 24.7 N (b) 3.09 N (c) 3.66 ms^{-1}
3. 384 N
4. 87.4 N, 1.63 ms^{-2}
5. 3.29
6. $0.497\mathbf{i} + 0.285\mathbf{j} - 0.532\mathbf{k}$, 0.782 ms^{-2}
7. 1550 N
8. 8.12, 3.74, 7.81; $10\mathbf{i} + 8\mathbf{j} - 5\mathbf{k}$; 13.7
9. (b) $93.2\mathbf{i} + 238.6\mathbf{j}$ (c) 2.54 ms^{-2}

Miscellaneous Exercises

1. (a) $-\mathbf{i} - 33\mathbf{j} + 7\mathbf{k}$ (b) $-6\mathbf{i} + 3\mathbf{j} - 17\mathbf{k}$
2. (a) 5 ms^{-2} (b) $\frac{1}{11.6}t^2(16\mathbf{i} - 21\mathbf{j} + 12\mathbf{k})$
3. 280 ms^{-2}
4. 3.18 ms^{-2}
5. (a) 6.35 ms^{-2} (b) 214 N
6. (a) 2.275 ms^{-2} (b) 4.55 m
7. 1958 N, 3880 N
8. 38.0°, 95 500 N, 4.78×10^{-3} ms^{-2}
9. $-88\mathbf{i} - 128\mathbf{j}$, 155 N
10. (a) 258 N (b) 11.2 N (c) 0.372 ms^{-2}
11. $80\mathbf{i}$, $-50\cos\alpha\,\mathbf{i} + 50\sin\alpha\,\mathbf{j}$, $-70\cos\beta\,\mathbf{i} - 70\sin\beta\,\mathbf{j}$;
    $\alpha = 60.0°$, $\beta = 38.2°$
12. 40.1°
13. 1.55 ms^{-2}, 16.2 ms^{-1}, 7.2 ms^{-1}
14. $10\sin\theta - 3\cos\theta$, 16.7°
15. 693 N
16. (a) 8 ms^{-2} (b) $\frac{5\pi}{4}$ $\left(\text{or } -\frac{3\pi}{4}\right)$
    (c) $(2\cos 2t - 2)\mathbf{i} + (2\sin 2t - 4t)\mathbf{j}$
17. (a) 1.5 (b) 0.24
18. (a) $(9\mathbf{i} + 17\mathbf{j})$ N
    (b) $\sqrt{370}$ N
    (c) $\frac{9}{\sqrt{370}}$
19. $-2\mathbf{i} + 3\mathbf{j}$, 124°
20. $R = (\lambda - 4)\mathbf{i} + 4\mathbf{j}$, $\lambda = 1$ or 7; $\frac{4}{3}$
21. $e^{-t}\left[-(\cos t + \sin t)\mathbf{i} + (\cos t - \sin t)\mathbf{j}\right]$;
    $2e^{-t}(\sin t\,\mathbf{i} - \cos t\,\mathbf{j})$
22. $v\left(\frac{8\mathbf{i}}{25} + \frac{44\mathbf{j}}{25}\right)$ km h^{-1}; $(5\mathbf{i} + 8\mathbf{j})$ km h^{-1}; 124°

5 PROJECTILES

Exercise 5A

1. 15 m, 17.3 m
2. 8.4 m
3. 49.8 ms^{-1}, 1.9° (below)
4. 18.8 m, 21.7 m

Exercise 5B

1. 34.6 m
2. 6.57 m
3. 23.9 ms^{-1}
4. Yes; with 50 cm to spare.
5. 3.12 m
6. 3 s, 52.0 m, 26.5 ms^{-1} at $49.1°$ to the horizontal.
7. The result holds for any angle of projection.

Exercise 5C

1. 1.8 m, 1.2 s, 12.5 m; 10.4 ms^{-1} at $5.5°$, 11.1 ms^{-1} at $21.1°$ downwards.
2. $24.3°$, $65.7°$; 1.65 s, 3.65 s; 40 m
3. 32.0 ms^{-1}, $38.7°$
4. 30 ms^{-1}, 4.24 s
5. $4.80°$, $85.20°$. The first is more likely.
6. 2.73 ms^{-1}. The locust must overcome air resistance.
7. $76.0°$

Miscellaneous Exercises

1. No. 2 m below top of wall.
2. $\psi \to -90°$
3. Yes. The critical angle of projection is $46.1°$.
8. The angles are $63.4°$ and $26.6°$.
9. 242 m
10. (a) 4.375 m (b) 48.3 ms^{-1}
 (c) 2.653 s or 0.204 s
11. $R = 245\sin 2\theta$; $\theta = \frac{\pi}{6}$
12. (a) $\dot{x} = u\cos\alpha$; $\dot{y} = u\sin\alpha - gt$

 (b) $x = ut\cos\alpha$; $y = ut\sin\alpha - \frac{1}{2}gt^2$

 Max. ht. $\frac{gT^2}{8}$; $\tan\alpha = \sqrt{2}$
13. Horiz. component $= \frac{4a}{T}$; vert. component $= \frac{gT}{2}$
14. Height of B $= \frac{v^2\sin^2\alpha}{2g}$; OC $= \frac{v^2\tan\alpha}{g\left(1+\frac{1}{4}\tan^2\alpha\right)}$;

 $\tan\alpha = \sqrt{2}$
15. $a = 82.5$, $b = 1.75$ (a) $\frac{7}{4}$

 (b) $\tan\alpha = 0.5$ or 3
16. (a) Vel. of P $= nu(3\mathbf{i} + 5\mathbf{j}) - gt\mathbf{j}$;

 Vel. of P relative to Q $= (3n+4)u\mathbf{i} + (5n-3)u\mathbf{j}$

 (b) $n = 4$ (c) $\frac{-12u^2}{g}$; $\frac{9u^2}{2g}$

6 WORK AND ENERGY

Exercise 6A

1. (a) 900 J (assuming running on the level)

 (b) 3.13×10^4 k J

 (c) 1.8×10^{-14} J

 (d) 1.8×10^5 J
2. 328 J
3. 250 N
4. 7.07 ms^{-1}
5. 6.32 ms^{-1}
6. 4510 J, 2.24 ms^{-1}
7. 2600 N
8. 1.28×10^7 J
9. The **useful** work is 123 J, 2.20 ms^{-1}
10. 540 N
11. 6.93 ms^{-1}

Exercise 6B

1. 1.2×10^6 J
2. 5×10^4 J
3. Assuming his centre of mass was 1 m above the ground at takeoff, 4960 J
4. (a) 800 J, $800 - 10x + \frac{v^2}{2}$ (relative to ground level)

 (c) 28.3 ms^{-1}, 40 ms^{-1}
5. (a) $\frac{1}{2}mv^2 + mgh$ (c) 11.8 m (d) 15.4 ms^{-1}

Exercise 6C

1. 20 kW
2. 2250 kW
3. Pull = 938 N; acceleration = 1.0625 ms^{-2}

4. 1080 N, 16.4 kW
5. 57 kW (1 knot = 0.515 ms^{-1})
6. 180 W
7. (a) 69.1 kW (b) 92.1 kW

Exercise 6D

1. 6.25 Nm^{-1}
2. 45 cm
3. (a) 62.5 Nm^{-1} (b) 0.625 kg

Exercise 6E

1. (a) 2 J (b) 6 J
2. 2 J
3. 1000 Nm^{-1}, 0.45 J
4. 0.1 m, 0.1 m, 15 J

Exercise 6F

2. (a) 1.29 ms^{-1} (b) 9.4 cm
3. 40.5 cm, 2.24 ms^{-1}
4. 7.67×10^4 N
5. $\dfrac{3mg}{2l}$
6. 6 m; (a) when $x = 0$ (b) when $x = 4$
7. (a) $\sqrt{\dfrac{800 - 70875x^2}{162}}$ (b) 10.6 cm

Exercise 6G

1. 13.
2. (a) 38, 12, 48 (b) $(9\mathbf{i} + 2\mathbf{j} - 9\mathbf{k})$, 98
3. (a) $\frac{1}{2}m\left(t^4 + t^2 + 1\right)$ (b) $m(2t\mathbf{i} + \mathbf{j})$

   (c) $mt\left(2t^2 + 1\right)$

Miscellaneous Exercises

1. 5.35 ms^{-1}
2. $d = \dfrac{(1 + 8l)^{\frac{1}{2}}}{4}$
3. 0.71 ms^{-1}
4. 0.95 m
5. 0.632 ms^{-1}
7. 2.5 ms^{-2}, $v = \frac{1}{2}\sqrt{gl}$, $a = 5\ \text{ms}^{-2}$
8. 0.127; without friction the deflection would be 0.141 m.
9. 1.5435 J; $v^2 = 10.16\ \text{m}^2\text{s}^{-2}$
10. a = 125 N; b = 0.75 Nm^{-2}s^2; 0.42 ms^{-2}; 4.9°
11. $\sqrt{2ga}$; $\sqrt{6ga}$
12. (a) 20 ms^{-1} (b) 11.44 ms^{-1}
13. Vel = $a(\omega - \omega\cos\omega t)\mathbf{i} + a(\omega - \omega\sin\omega t)\mathbf{j}$

    Acc = $a\omega^2\sin\omega t\,\mathbf{i} - a\omega^2\cos\omega t\,\mathbf{j}$

    (a) $ma\omega^2(\sin\omega t\,\mathbf{i} - \cos\omega t\,\mathbf{j})$ (b) $\dfrac{\pi}{4\omega}$

    (c) $ma^2\omega^2\left(\dfrac{3}{2} - \sqrt{2}\right)$; $\dfrac{5}{2}ma^2\omega^2$

    (d) $ma^2\omega^2\left(1 + \sqrt{2}\right)$
14. Max. extension = $\dfrac{l}{2}$
15. (a) $\dfrac{9}{4\sqrt{5}}\sqrt{(ga)}$ (b) $\dfrac{5a}{4}$
16. 2×10^8 J; 5.88×10^8 J; 9×10^8 J; 1.12×10^8 J
17. $\dfrac{g}{5}$; $\dfrac{12mg}{5}$ (a) $\dfrac{2v}{g}$ (b) $\dfrac{2v}{5}$

    (c) $\dfrac{21mv^2}{10}$ (d) $\dfrac{609mv^2}{250}$

7 CIRCULAR MOTION

Exercise 7A

1. (a) $\dfrac{1}{60}$ (b) $\dfrac{1}{10}$ (c) $\dfrac{\pi}{1800}$
2. (a) $\dfrac{2\pi}{\omega}$ (b) $2\pi r$ (c) $r\omega$
3. $2.6 \times 10^{-6}\ \text{s}^{-1}$, 0.92 km s^{-1}
4. $2.0 \times 10^{-7}\ \text{s}^{-1}$, 30 km s^{-1}

Exercise 7B

1. 2 s^{-1}, 0.9 ms^{-1}, 1.8 ms^{-2}
2. 17.3 s^{-1}, 6.93 ms^{-1}
3. (a) $v = 1\ \text{ms}^{-1}$ tangentially, $a = 1\ \text{ms}^{-2}$ radially inwards.

   (b) $v = 3t^2$ tangentially, $a = 6t$ tangentially, $9t^4$ radially inwards.
4. $0.864t$; 0.130 ms^{-2} tangentially, 0.448 ms^{-2} radially inwards.

Exercise 7C

1. 3.6 N, directed towards centre
2. 0.222, 1.19 *m* radially
3. 11.6 ms^{-1}
4. (a) 48 N (b) 33.6° (c) 1.66 m
5. 57.7 N, 1.70 ms^{-1}
6. 2.58 s^{-1}
7. 1 day, 4.23×10^{7} m, 3.63×10^{7} m, 3066 ms^{-1}; No

Exercise 7D

1. 15.8 ms^{-1} (56.9 kph), 44.6°
2. 44.7 ms^{-1}, 13.5 ms^{-1}
3. 41.0°
4. 25.2 ms^{-1}

Miscellaneous Exercises

1. (a) 1.05 rad s^{-1}

   (b) Lucy 2.1 ms^{-1}; Tom 1.57 ms^{-1}
2. 2 N
3. 581 N
4. 0.11
5. 0
6. $\sqrt{\dfrac{g}{3l}}$
7. $2\pi a\sqrt{\dfrac{a}{k}}$
8. $\dfrac{g}{l\omega^2}$
9. $\cos\theta = \dfrac{g}{2\pi^2}$
10. $\sqrt{\left(\dfrac{15ga}{4}\right)}$; 4 mg
11. $T_2 = 3mg\sec\theta$; $\dfrac{4}{3}$; $\dfrac{7g}{l\omega^2}$; $\omega \geq \sqrt{\left(\dfrac{7g}{l}\right)}$
12. 392 N (a) $\dfrac{\pi}{3}$; 8.85 rad s^{-1}

    (b) 11.4 rad s^{-1}
13. $mg + ml\omega^2$; $ml\omega^2 - mg$; $\sqrt{\dfrac{6l}{g}}$

8 SIMPLE HARMONIC MOTION

Exercise 8A

1. No effect
2. 7.70 s
4. Reduce *l* by 1.37%
5. Same period, amplitudes different
6. $\theta = \dfrac{\pi}{36}\cos\left(\sqrt{50}t\right)$; $\theta = 0.36\cos\left(\sqrt{50}t - 1.33\right)$

Exercise 8B

1. $v = 0.3$ ms^{-1}
2. (a) $v = \sqrt{2gl\left(1-\cos\theta_0\right)}$

   (b) $v = \sqrt{u^2 + 2gl\left(1-\cos\theta_0\right)}$

Exercise 8C

1. $v = 240\cos\left(100\pi t + \alpha\right)$
2. $x = 0.01\cos\left(400t + \alpha\right)$
3. 0.712 ms^{-1} to 2.85 ms^{-1}
4. $P = 25\cos\left(6t + \alpha\right) + 95$
5. $h = 1.5 + 0.5\sin\left(105t\right)$

Exercise 8D

1. 63 g; small angle
2. (a) $x = 0.03\cos\left(\cos\left(\sqrt{\dfrac{1}{800}}\right)t\right) + 0.4125$ from top position

   (b) 178 s (c) 0.03 m
3. 44.7 Nm^{-1}
4. Same period, different amplitudes, 0.02 m, 0.03 m
5. $\dfrac{2\pi}{\sqrt{30}}$ s; 0.02 m
6. no change
7. $v = 5.48$ ms^{-1}
8. 0.256 J; 4.53 ms^{-1}; $v = 4.41$ ms^{-1}
9. 6.3 ms^{-2}
10. (a) 0.5 m (b) 1.4 s; 0.3 m
11. 2.01 ms^{-1}; 505 ms^{-2}

Miscellaneous Exercises

1. (a) $\frac{g}{\pi^2} \approx 1.01$ m (b) 1.4 mm shorter
2. 75%
3. $\pi\sqrt{2}$
4. 0.125 ms^{-1}
5. (a) $8\sqrt{3}$ ms^{-1} (b) $\frac{\pi}{48}$ s
6. 0.942 ms^{-1}; 0.816 ms^{-1} 0.911 m
7. π; 13 cm; 0.12 s
8. $a = 13$ m; $\omega = \frac{\pi}{8}$ rad s^{-1}; $\phi = -1.18$ rad
9. $\sqrt{\frac{ag}{8}}$
10. (a) 0.5 m (b) $\frac{\pi}{15}$ s (c) 15 ms^{-1}

    (d) 0.0309 s; $\lambda = 360$ N, 14.4 J
11. $\frac{5}{3\pi}$ m; 0.25 m; $\frac{1}{18}$ s
12. (a) 4π s (b) 15 m (c) 7.5 ms^{-1}; $\frac{\pi}{3}$ s

9 PHYSICAL STRUCTURES

Exercise 9A

1. (a) 10.4 N at 90° to the 6 N force

   (b) 7.7 N at 140° to either 5 N force
2. (a) $X = 10.4$, $Y = 6$ (b) $A = 3.4$, $B = 5.8$

   (c) $C = 7.5$ $D = 2.2$ (d) $P = 2.9$, $Q = 5.2$

Exercise 9B

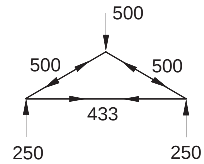
1. (a) Tension of 238 N in PQ, thrust of 311 N in QR, force on wall at P of 238 N away from wall in the direction PQ, force on wall at R of 311 N towards wall in the direction QR.

   (b) Tension of 566 N in PQ, thrust of 400 N in QR, force on wall at P of 566 N away from wall in the direction PQ, force on wall at R of 400 N towards wall in the direction QR.

   (c) Tension of 100 N in PQ, thrust of 100 N in QR, force on wall at P of 100 N away from wall in direction PQ, force on wall at R of 100 N towards wall in direction QR.
2. Thrusts of 354 N in PQ and QR, tension of 250 N in PR, normal contact forces of 250 N at P and R.

Exercise 9C

1.



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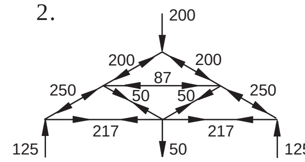

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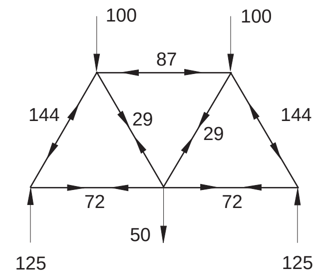

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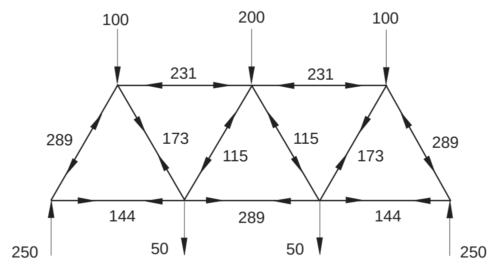

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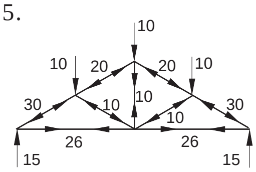

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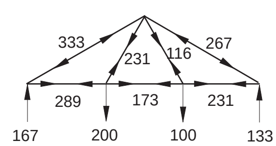

Exercise 9D

1. 1.5 m to the right of the pivot
2. (a) Tension in AC = 1600 N

   Thrust in BC = 1385.6 N

   (b) AC

   (c) 1385.6 N
3. Tension in BC = 35863.0 N

   Thrust in AB = 29282.0 N
4. Tension in BC = 26897.3 N

   Thrust in AB = 21961.5 N

   Note how the forces change when the mass is suspended at different points.
5. (a) 1.6 m to the right of the pivot

   (b) He has to move approximately 2 cm which is not significant

   (c) Yes (d) No (e) $y = 2x - 1.2$

Exercise 9E

1. 21 Nm
2. 6.06 al
3. 13 Nm

Exercise 9F

1. (a) 2.5 N
   (b) 0.6 m from the end where the 50 g is suspended.
2. Tension in the string suspended between 50 g mass and the centre is 0.8 N. Tension in other string is 1.7 N.
3. 1.05 kg
4. 120 g, 300 g
5. (a) at B (b) 700 g
6. The 2 kg mass can lie 20 cm either side of the centre of the plank.
7. (a) 447 N
   (b) horizontal component 200 N, vertical component 250 N downwards
8. (a) 35.5 N
   (b) the reaction at the hinge has horizontal component 15.0 N and vertical component 67.9 N.

Exercise 9G

1. (a) 0.625 N from 3 N weight
   (b) 1.27 m from 4 N weight
   (c) 6 m from 5 N weight
2. 1.25 m; 8 N
3. 2.06 m

Exercise 9H

1. (a) On the line of symmetry 3.14 from O
   (b) On the line of symmetry 0.92 m from AB
   (c) On the line of symmetry 1.7 m from AB
2. (a) 2.6 m (b) 2.04 m
3. Mass is 10.8 kg; centre of gravity is 1.33 m from AD; tension is 596 N; horizontal and vertical components of the contact force are 516 N, 310 N.

Exercise 9I

1. (a) 2 N (b) 4 N; 0.4
2. 2.5 N, 4.44 N
3. 6 m
4. 26.57°, 18.43°, 5.71°
5. 60°
6. To $\frac{13}{14}$ths of its length

Miscellaneous Exercises

1. 5 kg
2. 27 cm from centre of disc
3. 8.7 cm
4. $\frac{5h}{8}$
5. 7 kg
6. (a) 60 N; 80 N (b) $\frac{120}{\sqrt{3}}$ N; $\frac{60}{\sqrt{3}}$ N; $\frac{70}{\sqrt{3}}$ N
7. $\frac{a}{2}\left(54\right)^{\frac{1}{4}}$
8. 25 N; thrust $\frac{50}{\sqrt{3}}$ N in AC and BC;
   tension $\frac{25}{\sqrt{3}}$ N in AB
9. (a) $0.25a$ (b) $\lambda = 2$
10. $\frac{45a}{56}$ from base
11. Pa anticlockwise; $P\sqrt{85}$
12. $\frac{40}{3}$; $\frac{80}{3}$; 20 N
14. (a) $\sqrt{10}$ (b) 71.6° (c) 4 m
15. 25 N; $50\sqrt{3}$ N
16. $\frac{5a}{6}$, $\frac{5a}{6}$; (b) $m = \frac{M}{2}$ (c) $P = 2Mg$
17. (a) $\frac{Wa\cos\theta}{2h}$, $\frac{W}{2}\left(1+\frac{a\sin\theta}{h}\right)$; $\frac{a}{h}$ (b) $\frac{W\sqrt{3}}{4}$
18. $Mg\cos\alpha$; $Mg\sin\alpha - mg$; $2-\sqrt{3}$

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