FIRRHILL HIGH SCHOOL ADVANCED HIGHER MATHEMATICS: REVISION NOTES

PARTIAL FRACTIONS: REVISION POINTS

Rational functions that have at most a cubic denominator will be studied. The functions will factorise into either (a) **distinct linear factors**, (b) a linear factor and an irreducible **quadratic factor** or (c) **repeated linear factors.**

(a) $\frac{x+1}{(x+3)(x-2)} = \frac{A}{(x+3)} + \frac{B}{(x-2)}$ etc.

(b) $\frac{x+1}{(x+3)(x^2-2x+3)} = \frac{A}{(x+3)} + \frac{Bx+C}{(x^2-2x+3)}$ etc.

(c) $\frac{x+1}{(x+3)(x-2)^2} = \frac{A}{(x+3)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$ etc.

Remember to use long division if the degree of the numerator is the same or larger than the degree of the denominator.

These revision points are explained more fully below with worked examples.

Partial Fractions: Distinct Linear Factors

Writing $\frac{5x-1}{(x+1)(x-2)}$ as $\frac{2}{x+1}+\frac{3}{x-2}$ is known as writing $\frac{5x-1}{(x+1)(x-2)}$ in **partial fractions**.

Note that the denominator of $\frac{5x-1}{(x+1)(x-2)}$ contains **distinct linear factors** and that the partial fractions are of the form $\frac{A}{x+1}+\frac{B}{x-2}$ for some constants *A* and *B*.

Worked Example 1

Express $\frac{x+7}{(x-2)(x+1)}$ in partial fractions.

Solution

Note that the denominator contains **distinct linear factors**.

$$\frac{x+7}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}$$
$$=\frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

Hence $x+7=A(x+1)+B(x-2)$.

To eliminate the first bracket and find the value of *B*, put $x=-1$:

$$6=A(0)+B(-3)$$
$$\Rightarrow \quad -3B=6$$
$$\Rightarrow \quad B=-2$$

To eliminate the second bracket and find the value of *A*, put $x=2$:

$$9=A(3)+B(0)$$
$$\Rightarrow \quad 3A=9$$
$$\Rightarrow \quad A=3$$

Hence $\frac{x+7}{(x-2)(x+1)}=\frac{3}{x-2}-\frac{2}{x+1}$.

Partial Fractions: Repeated Linear Factors

If the denominator contains a **repeated linear factor**, more than one partial fraction must be included for this factor, as illustrated in the example below.

Worked Example 2

Express $\dfrac{x^2-7x+9}{(x+2)(x-1)^2}$ in partial fractions.

Solution

Note that the denominator contains a **repeated linear factor**.

$$\frac{x^2-7x+9}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$
$$= \frac{A(x-1)^2 + B(x+2)(x-1) + C(x+2)}{(x+2)(x-1)^2}$$

Hence $x^2-7x+9 = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$.

Put $x=1$ $\Rightarrow$ $3 = A(0)^2 + B(3)(0) + C(3)$
$\Rightarrow$ $3C = 3$
$\Rightarrow$ $C = 1$

Put $x=-2$ $\Rightarrow$ $27 = A(-3)^2 + B(0)(-3) + C(0)$
$\Rightarrow$ $9A = 27$
$\Rightarrow$ $A = 3$

To find the value of *B*, we must substitute a third number for *x*. It is convenient to use a simple number, such as $x=0$.

Put $x=0$ $\Rightarrow$ $9 = A(-1)^2 + B(2)(-1) + C(2)$
$\Rightarrow$ $A - 2B + 2C = 9$
$\Rightarrow$ $3 - 2B + 2 = 9$
$\Rightarrow$ $5 - 2B = 9$
$\Rightarrow$ $-2B = 4$
$\Rightarrow$ $B = -2$

Hence $\dfrac{x^2-7x+9}{(x+2)(x-1)^2} = \dfrac{3}{x+2} - \dfrac{2}{x-1} + \dfrac{1}{(x-1)^2}$.

Partial Fractions: Irreducible Quadratic Factors

If the denominator contains an **irreducible quadratic factor** $q(x)$, the partial fraction corresponding to this factor is of the form $\frac{Bx+C}{q(x)}$.

The discriminant can be used to verify that a quadric factor cannot be factorised.

Worked Example 3

Express $\frac{3x^2+2x+1}{(x+1)(x^2+2x+2)}$ in partial fractions.

Solution

We must first use the discriminant to verify that the quadratic factor x^2+2x+2 is **irreducible.**

For x^2+2x+2: $a=1$, $b=2$, $c=2$

$$b^2-4ac = 2^2-4\times1\times2$$
$$= -4$$

$b^2-4ac<0$, so x^2+2x+2 is irreducible.

$$\frac{3x^2+2x+1}{(x+1)(x^2+2x+2)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+2x+2}$$
$$= \frac{A(x^2+2x+2)+(Bx+C)(x+1)}{(x+1)(x^2+2x+2)}$$

Hence $3x^2+2x+1 = A(x^2+2x+2)+(Bx+C)(x+1)$.

Put $x=-1$ $\Rightarrow$ $2 = A(1)+0$
$\Rightarrow$ $A=2$

To find the values of *B* and *C*, we must substitute two other numbers for *x*. It is convenient to use simple numbers, such as $x=0$ and $x=1$.

Put $x=0$ $\Rightarrow$ $1 = A(2)+(C)(1)$
$\Rightarrow$ $2A+C=1$
$\Rightarrow$ $4+C=1$
$\Rightarrow$ $C=-3$

Put $x=1$ $\Rightarrow$ $6 = A(5)+(B+C)(2)$
$\Rightarrow$ $5A+2B+2C=6$
$\Rightarrow$ $10+2B-6=6$

$\Rightarrow \quad 4+2B=6$
$\Rightarrow \quad 2B=2$
$\Rightarrow \quad B=1$

Hence $\dfrac{3x^2+2x+1}{(x+1)(x^2+2x+2)}=\dfrac{2}{x+1}+\dfrac{x-3}{x^2+2x+2}$.

ALGEBRAIC LONG DIVISION

The algebraic fraction $\dfrac{x+1}{x^2+2}$ is an example of a proper rational function, since the degree of the numerator is less than the degree of the denominator.

The algebraic fraction $\dfrac{x^2+2}{x+1}$ is an example of an **improper rational function**, since the degree of the numerator is greater than or equal to the degree of the denominator.

An improper rational function can be expressed as the sum of a polynomial and a proper rational function using **algebraic long division**.

Worked Example 4

Express $\dfrac{x^2+2x+4}{x+1}$ as the sum of a polynomial function and a proper rational function.

Solution

Set up the division as below.

$$x+1\,\overline{)\,x^2+2x+4}$$

Consider the first term of the divisor $x+1$ (x) and the first term of x^2+2x+4 (x^2).
To change x into x^2 you multiply by x.
You can now write x in the appropriate position at the top of the division as below.

$$\begin{array}{r} x \\ x+1\,\overline{)\,x^2+2x+4} \end{array}$$

Now multiply the divisor $(x+1)$ by the factor x and write the answer as below.

$$
\begin{array}{r|l}
 & \quad\; x \\ \hline
x+1 & x^2+2x+4 \\
 & x^2+\;\; x
\end{array}
$$

Now subtract (x^2+x) from (x^2+2x+4) as below.

$$
\begin{array}{r|l}
 & \quad\; x \\ \hline
x+1 & x^2+2x+4 \\
 & x^2+\;\; x \\ \cline{2-2}
 & \quad\;\; x+4
\end{array}
$$

The steps carried out so far are then repeated.

Consider the first term of the divisor $x+1$ (x) and the first term of $x+4$ (x).
To change x into x you multiply by 1.
You can now write 1 in the appropriate position at the top of the division as below.

$$
\begin{array}{r|l}
 & \quad\; x+1 \\ \hline
x+1 & x^2+2x+4 \\
 & x^2+\;\; x \\ \cline{2-2}
 & \quad\;\; x+4
\end{array}
$$

Now multiply the divisor $(x+1)$ by the factor 1 and write the answer as below.

$$
\begin{array}{r|l}
 & \quad\; x+1 \\ \hline
x+1 & x^2+2x+4 \\
 & x^2+\;\; x \\ \cline{2-2}
 & \quad\;\; x+4 \\
 & \quad\;\; x+1
\end{array}
$$

Now subtract $(x+1)$ from $(x+4)$ as below.

$$
\begin{array}{r|l}
 & \quad\; x+1 \\ \hline
x+1 & x^2+2x+4 \\
 & x^2+\;\; x \\ \cline{2-2}
 & \quad\;\; x+4 \\
 & \quad\;\; x+1 \\ \cline{2-2}
 & \qquad\quad\; 3
\end{array}
$$

The process cannot now be continued and the answer can be read from the division as below.

$$\frac{x^2 + 2x + 4}{x + 1} = x + 1 + \frac{3}{x + 1}$$

Worked Example 5

Express $\frac{x^3 - 2x + 5}{x^2 + 2x - 3}$ as the sum of a polynomial and a proper rational function.

Solution

$$\begin{array}{r|l} & \quad\quad\quad\;\; x - 2 \\ \hline x^2 + 2x - 3 & x^3 + 0x^2 - 2x + 5 \\ & x^3 + 2x^2 - 3x \\ \cline{2-2} & \quad -2x^2 + x + 5 \\ & \quad -2x^2 - 4x + 6 \\ \cline{2-2} & \quad\quad\quad\;\; 5x - 1 \end{array}$$

Hence $\frac{x^3 - 2x + 5}{x^2 + 2x - 3} = x - 2 + \frac{5x - 1}{x^2 + 2x - 3}$.

PARTIAL FRACTIONS FOR IMPROPER RATIONAL FUNCTIONS

An improper rational function can be expressed as the sum of a polynomial and partial fractions.

Worked Example 6

Express $\frac{x^3 - 3x}{x^2 - x - 2}$ as the sum of a polynomial and partial fractions.

Solution

$$
\begin{array}{r|l}
 & \qquad\quad x + 1 \\
\hline
x^2 - x - 2 & x^3 + 0x^2 - 3x + 0 \\
 & x^3 - \ \ x^2 - 2x \\
\hline
 & \qquad x^2 - \ \ x + 0 \\
 & \qquad x^2 - \ \ x - 2 \\
\hline
 & \qquad\qquad\qquad 2
\end{array}
$$

Hence $\dfrac{x^3 - 3x}{x^2 - x - 2} = x + 1 + \dfrac{2}{x^2 - x - 2}$ ...(*)

Now express $\dfrac{2}{x^2 - x - 2}$ in partial fractions.

Note that $\dfrac{2}{x^2 - x - 2} = \dfrac{2}{(x-2)(x+1)}$ and that the denominator **contains distinct linear factors**.

$$
\begin{aligned}
\frac{2}{(x-2)(x+1)} &= \frac{A}{x-2} + \frac{B}{x+1} \\
&= \frac{A(x+1) + B(x-2)}{(x-2)(x+1)}
\end{aligned}
$$

Hence $2 = A(x+1) + B(x-2)$.

Put $x = -1$ $\Rightarrow$ $2 = A(0) + B(-3)$

$\Rightarrow$ $-3B = 2$

$\Rightarrow$ $B = -\dfrac{2}{3}$

Put $x = 2$ $\Rightarrow$ $2 = A(3) + B(0)$

$\Rightarrow$ $3A = 2$

$\Rightarrow$ $A = \dfrac{2}{3}$

Hence $\dfrac{2}{x^2 - x - 2} = \dfrac{\frac{2}{3}}{x-2} - \dfrac{\frac{2}{3}}{x+1} = \dfrac{2}{3(x-2)} - \dfrac{2}{3(x+1)}$.

From (*): $\dfrac{x^3 - 3x}{x^2 - x - 2} = x + 1 + \dfrac{2}{3(x-2)} - \dfrac{2}{3(x+1)}$.

Examples for you to try:

1. Find partial fractions for :-

(a) $\frac{2x-5}{(x+2)(x-1)}$ (b) $\frac{2x}{x^2-1}$ (c) $\frac{x+4}{x(x+2)}$

(d) $\frac{x-8}{2x^2+3x-2}$ (e) $\frac{5x+1}{2x^2+5x-3}$ (f) $\frac{7}{3x^2+5x-2}$

2. Find partial fractions for :-

(a) $\frac{3x+1}{(x+1)(x^2+1)}$ (b) $\frac{x-4}{x(x^2+2)}$ (c) $\frac{5+x}{(1-x)(5+x^2)}$

(d) $\frac{x^2-10}{(2x-1)(x^2+3)}$ (e) $\frac{x^2-13}{(x+2)(x-1)^2}$ (f) $\frac{3x^2+7x+1}{x^3+2x^2+x}$

Find partial fractions for the following (remember to divide if necessary) :–

(a) $\frac{x^2+3x}{x^2-4}$ (b) $\frac{x^2-x+1}{x^2-x-2}$ (c) $\frac{x^2}{(x-1)^2}$

(d) $\frac{x^3-x^2+x+1}{x(x^2+1)}$ (e) $\frac{x^3}{x^2-4}$ (f) $\frac{x^3+3x^2+10}{x^2+5x+4}$

Solutions:

1 (a) $\frac{3}{x+2}-\frac{1}{x-1}$ (b) $\frac{1}{x+1}+\frac{1}{x-1}$ (c) $\frac{2}{x}-\frac{1}{x+2}$

(d) $\frac{2}{x-2}-\frac{3}{2x+1}$ (e) $\frac{2}{x+3}+\frac{1}{2x-1}$ (f) $\frac{3}{3x-1}-\frac{1}{x+2}$

2 (a) $\frac{x+2}{x^2+1}-\frac{1}{x+1}$ (b) $\frac{2}{x}-\frac{2x-1}{x^2+2}$ (c) $\frac{1}{1-x}+\frac{x}{5+x^2}$

(d) $\frac{2x+1}{x^2+3}-\frac{3}{2x-1}$ (e) $\frac{2}{x-1}-\frac{4}{(x-1)^2}-\frac{1}{x+2}$ (f) $\frac{1}{x}+\frac{2}{x+1}+\frac{3}{(x+1)^2}$

(a) $1+\frac{5}{2(x-2)}+\frac{1}{2(x+2)}$ (b) $1+\frac{1}{x-2}-\frac{1}{x+1}$

(c) $1+\frac{2}{x-1}+\frac{1}{(x-1)^2}$ (d) $1-\frac{1}{x}+\frac{2x}{x^2+1}$

(e) $x+\frac{2}{x+2}+\frac{2}{x-2}$ (f) $x-2+\frac{4}{x+1}+\frac{2}{x+4}$