WORKED EXAMPLE – TEST PAPER D

1. By Pythagoras' Theorem

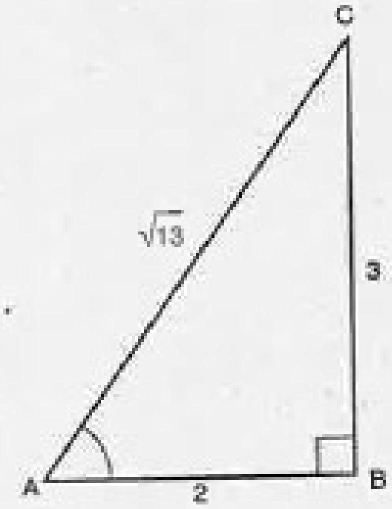
$AC^2 = 3^2 + 2^2 = 13$

$\cos A = \dfrac{2}{\sqrt{13}}$

rationalise denominator $\dfrac{2}{\sqrt{13}} \times \dfrac{\sqrt{13}}{\sqrt{13}} = \dfrac{2\sqrt{13}}{13}$

$\cos A = \dfrac{2\sqrt{13}}{13} = \dfrac{2}{13}\sqrt{13}$

In form $p\sqrt{13} \Rightarrow p = \dfrac{2}{13}$



2. $f(x) = \sin^3 x + \cos^2 x$

$\sin^3 x = (\sin x)^3$

By chain rule method if $f(x) = (\sin x)^3$

then $f'(x) = 3(\sin x)^2 \cos x$

$= 3 \cos x \sin^2 x$

and $\cos^2 x = (\cos x)^2$

By chain rule method if $f(x) = (\cos x)^2$

then $f'(x) = 2(\cos x)^1 (-\sin x)$

$= -2 \cos x \sin x$

Hence $f(x) = \sin^3 x + \cos^2 x$

$\Rightarrow f'(x) = 3 \cos x \sin^2 x - 2 \cos x \sin x$

Which factorised $= \sin x \cos x\,(3 \sin x - 2)$

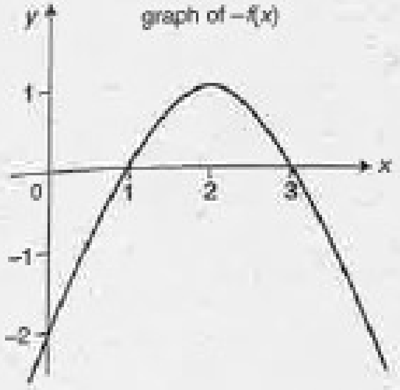
$f'\left(\dfrac{\pi}{4}\right) = \sin\dfrac{\pi}{4} \cos\dfrac{\pi}{4}\left(3 \sin\dfrac{\pi}{4} - 2\right)$ Note: $\sin\dfrac{\pi}{4} = \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

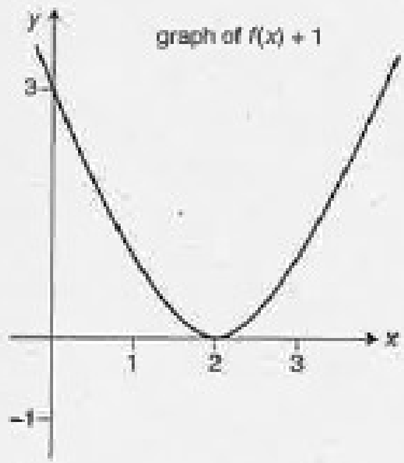
$= \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{\sqrt{2}}\left(\dfrac{3}{\sqrt{2}} - 2\right)$

$= \dfrac{1}{2}\left(\dfrac{3}{\sqrt{2}} - 2\right)$

$= \dfrac{3}{2\sqrt{2}} - 1$ or $\dfrac{3 - 2\sqrt{2}}{2\sqrt{2}}$

3. Graph of $-f(x)$ Graph of $f(x)+1$





4. *(a)* $0 \le x \le 360$

$f: x \rightarrow \sin 4x$ cuts x-axis when $\sin 4x = 0$

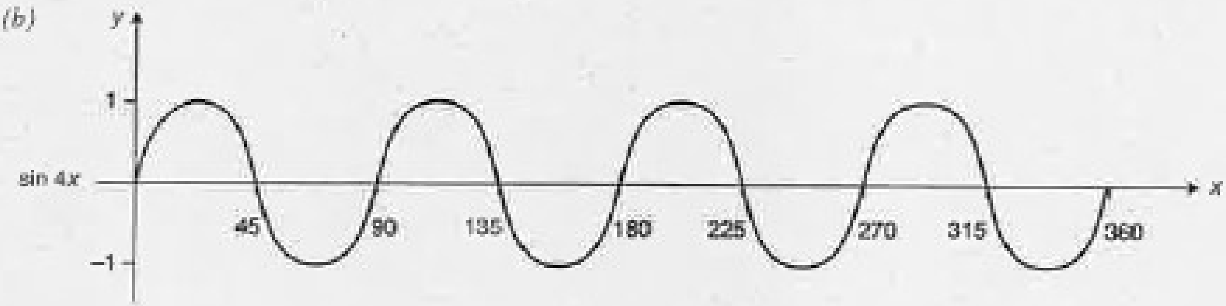
period $= \dfrac{360}{4} = 90$ $4x = 0, 180$

repeats 4 times in 360° $x = 0, 45°$

$0, 90+0, 180+0, 270+0, 45, 90+45, 180+45, 270+45$

$x = \{0, 45, 90, 135, 180, 225, 270, 315, 360\}$ 9 times

(b)



5. *(a)* Line which makes an angle of 45° with the positive direction of the x-axis has a tangent of 1.

Hence $m = 1$

Line with gradient = 1

Passes through point $(0, -1)$

$y - b = m(x - a)$

$y - (-1) = 1(x - 0)$

$y + 1 = x$ or $y = x - 1$ is required equation.

(b) If a line is perpendicular to another line then the product of their gradients = −1.

Hence perpendicular line has gradient = −1

Since $1 \times (-1) = -1$

Line with gradient −1 passes through point $(1, 3)$.

$y - b = m(x - a)$

$y - 3 = -1(x - 1)$

$y - 3 = -x + 1$

$y + x = 4$

(c) Perpendicular line has y-intercept $(0, 4)$ and since $m = -1$ angle made is 135°.

6. *(a)* $A = 3\cos\left(x - \frac{\pi}{6}\right)$

$$\begin{aligned}
\text{Maximum value} &= 3 \\
\text{when } \left(x - \frac{\pi}{6}\right) &= 0 \\
\text{since } \cos 0 &= 1 \\
\Rightarrow \quad x - \frac{\pi}{6} &= 0 \\
\Rightarrow x &= \frac{\pi}{6}
\end{aligned}$$

$$\begin{aligned}
\text{Minimum value} &= -3 \\
\text{when } \quad x - \frac{\pi}{6} &= \pi \\
\text{since } \cos \pi &= -1 \\
\Rightarrow \quad x - \frac{\pi}{6} &= \pi \\
x &= \pi + \frac{\pi}{6} \\
&= \frac{7\pi}{6}
\end{aligned}$$

Maximum value = 3, minimum value = −3.

(b) Maximum turning point $\left(\frac{\pi}{6}, 3\right)$

Minimum turning point $\left(\frac{7\pi}{6}, -3\right)$

7. A(1, 2), B(−3, 4), C(5, 6)

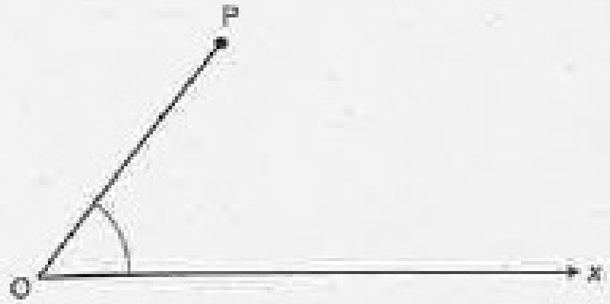
$\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\boldsymbol{b} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ $\boldsymbol{c} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

We find the position vector of the centroid by:

$$\begin{aligned}
\text{centroid} &= \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) \\
&= \frac{1}{3}\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix}\right] \\
&= \frac{1}{3}\begin{pmatrix} 3 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \text{position vector}
\end{aligned}$$

Hence coordinates of centroid = (1, 4)

8. P(2, –1, 3)



The unit vector parallel to x-axis $= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$p = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ |p| = 2^2 + 1^2 + 3^2 = \sqrt{14}, |x| = 1$$

$$\cos P\hat{O}X = \frac{p \cdot x}{|p||x|} = \frac{\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{(\sqrt{14})(1)} = \frac{2+0+0}{\sqrt{14}} = \frac{2}{\sqrt{14}}$$

$$\cos P\hat{O}X = \frac{2}{\sqrt{14}}$$

$$P\hat{O}X = 57{\cdot}7°$$

9. $u_{r+1} = mu_r + c$, $\quad u_0 = -1, u_1 = 7$ and $u_2 = -9$.

(a) $u_1 = m(u_0) + c \Rightarrow 7 = m(-1) + c$ ①

$u_2 = m(u_1) + c \Rightarrow -9 = m(7) + c$ ②

② – ① – 16 $= 8m \Rightarrow m = -2$

Substitute $m = -2$ in ① $7 = -2(-1) + c$

$7 = 2 + c \Rightarrow c = 5$

Recurrence relation is $u_{r+1} = -2u_r + 5$

(b) $u_3 = -2u_2 + 5$

$= -2(-9) + 5$

$u_3 = 23$

u_{-1} is found by using u_0

$u_0 = -2u_{-1} + 5$

$-1 = -2u_{-1} + 5$

$-6 = -2u_{-1}$

$\frac{-6}{-2} = u_{-1}$

$3 = u_{-1}$

$u_{-1} = 3$

(c) $u_r = u_{r+1}$

$\Rightarrow u_r = -2u_r + 5$

$\Rightarrow 3u_r = 5$

$u_r = \frac{5}{3}$

Test $-2\left(\frac{5}{3}\right) + 5 = \frac{5}{3}$

10. $g(x) = 3 - x^2$

$f(x) = 1 - 2x$

$$g(f(x)) = g(1-2x) = 3 - (1-2x)^2$$
$$= 3 - (1 + 4x^2 - 4x)$$
$$= 3 - 1 - 4x^2 + 4x$$
$$= 2 + 4x - 4x^2$$
$$g(f(x)) = 2(1 + 2x - 2x^2)$$

11. *(a)* $f(x) = 3x^2 - 2x + 5$

using the discriminant $b^2 - 4ac$

If $b^2 - 4ac < 0$

$f(x)$ has no real roots.

$a = 3, b = -2, c = 5$

$b^2 - 4ac$

$(-2)^2 - 4(3)(5)$

$4 - 60 = -56$

$\sqrt{-56} \in R$ $f(x)$ has no real roots.

(b) Completing the square

$3x^2 - 2x + 5$

$$= 3\left(x^2 - \frac{2}{3}x + \frac{5}{3}\right)$$
$$= 3\left[\left(x - \frac{2}{6}\right)^2 - \left(\frac{2}{6}\right)^2 + \frac{5}{3}\right]$$
$$= 3\left[\left(x - \frac{1}{3}\right)^2 - \frac{1}{9} + \frac{5}{3}\right]$$
$$= 3\left[\left(x - \frac{1}{3}\right)^2 + \frac{14}{9}\right]$$
$$= 3\left(x - \frac{1}{3}\right)^2 + \frac{14}{3}$$

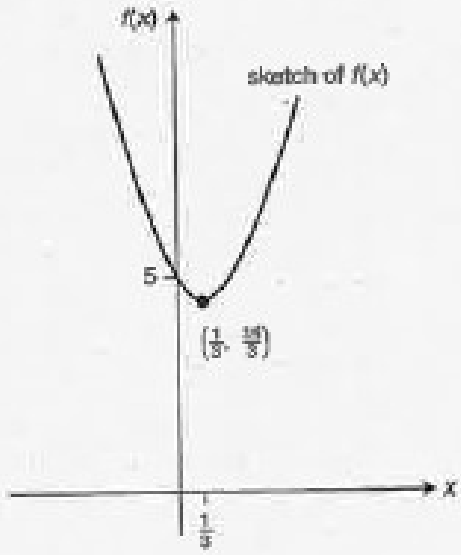
$\Rightarrow$ minimum value $= \dfrac{14}{3}$ when $x = \dfrac{1}{3}$

Since $\left(x - \dfrac{1}{3}\right)^2$ is never negative.

Sketch of $3x^2 - 2x + 5$ cuts y-axis at (0, 5).

Minimum T.P. $\left(\dfrac{1}{3}, \dfrac{14}{3}\right)$.

(c) Sketch of $f(x)$



12. *(a)* $y = x^3 + 2x^2 - 4$

The point of contact is $(-1, f(-1)) = (-1, -3)$

$f(x) = x^3 + 2x^2 - 4;$ Gradient $= f'(x) = 3x^2 + 4x;$ $m = -1$

Gradient $= f'(-1) = 3(-1)^2 + 4(-1);$ $m = -1$

$m = -1$ through $(-1, -3)$;
$$\begin{aligned} y - b &= m(x - a) \\ y - (-3) &= -1(x - (-1)) \\ y + 3 &= -x - 1 \\ y + x &= -4 \text{ is the required equation.} \end{aligned}$$

(b) $y = -x - 4;$ $m = -1;$ the tangent of the angle $= -1$

$\tan^{-1}(1) = 45°$ in quadrant 1

Since $m = -1$, the angle is in quadrant 2 and

is $180° - 45° = 135°$ (see diagram)