WORKED EXAMPLE – TEST PAPER C

1. $y = 5\log_2(2x+2)$
   At B, $x = 0$
   $y = 5\log_2(0+2)$
   $y = 5\log_2(2)$
   $y = 5 \times 1 = 5$
   coordinates of B(0, 5)

   $y = 5\log_2(2x+2)$
   At C, $y = 10$
   $= 5\log_2(2x+2) = 10$
   $\log_2(2x+2) = 2$
   $2^2 = 2x+2$
   $2x+2 = 4$
   $2x = 2;\ x = 1$
   coordinates of C(1, 10)

   $y = 5\log_2(2x+2)$
   At A, $y = 0$
   $5\log_2(2x+2) = 0$
   $\log_2(2x+2)^5 = 0$
   $2^0 = (2x+2)^5$
   $(2x+2)^5 = 1$
   $2x+2 = 1$
   $2x = -1;\ x = -\frac{1}{2}$
   coordinates of A $\left(-\frac{1}{2}, 0\right)$

2. $f(x) = 2x^3 + mx$
   $f'(x) = 6x^2 + m$ (the gradient of the tangent)
   $f'(2) = 6(2)^2 + m$ (Note $f'(-2) = f'(2)$)
   The gradient of the tangent = 0 for stationary values.
   Hence $f'(2) = 0$
   $\Rightarrow 24 + m = 0$
   $m = -24$
   So $f(x) = 2x^3 - 24x$
   and $f(-1) = 2(-1)^3 - 24(-1)$
   $= -2 + 24$
   $= 22$

3. *(a)* Circle $x^2 + y^2 - 2x + 6y + 1 = 0$
   General equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$
   where centre $= (-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$
   hence centre $= (1, -3)$, radius $= \sqrt{1^2 + 3^2 - 1} = 3$

   (b) Centre $(1, -3)$ reflected in y-axis $= (-1, -3)$
   $-g = -1$, hence $2g = 2$, f unchanged, r unchanged.
   Equation of circle after reflection in the y-axis is
   $$x^2 + y^2 + 2x + 6y + 1 = 0$$

4. A(−1, 3), B(2, −1), C(5, 4)

$m_{AC}\ \dfrac{4-3}{5-(-1)} = \dfrac{1}{6}$ (the gradient of line AC)

If AC is perpendicular to BQ

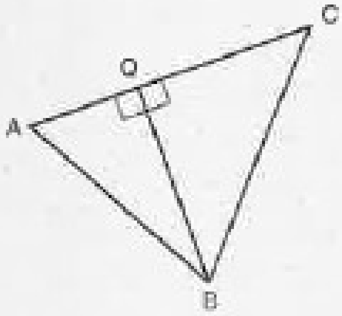
then $m_{AC} \times m_{BQ} = -1$

$m_{AC} = \dfrac{1}{6}$ hence $m_{BQ} = -6$

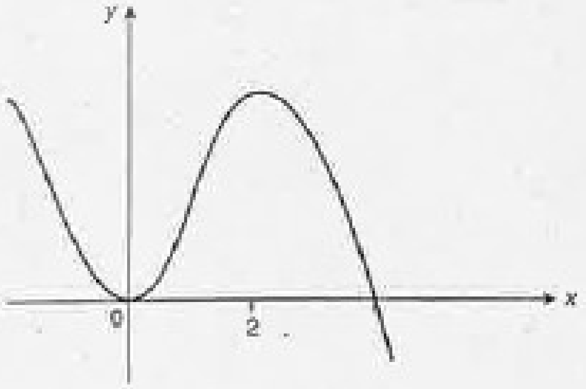
$m_{BQ} = -6$ through B(2, −1)

$$y - b = m(x - a)$$
$$y + 1 = -6(x - 2)$$
$$y + 1 = -6x + 12$$
$$y = -6x + 11$$

Equation of BQ is $y + 6x = 11$



5. Graph of $f(x)$.



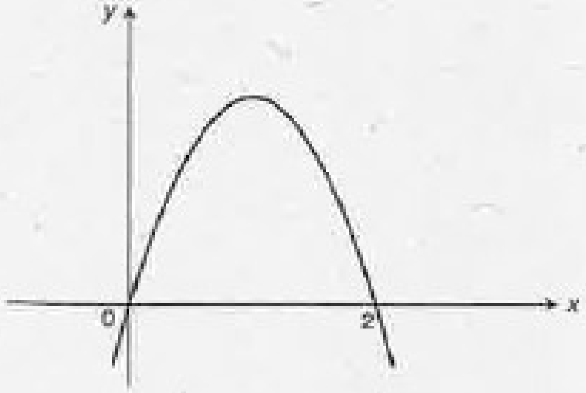
A cubic function $ax^3 + bx + c$

$f'(x)$ will be of form $3ax^2 + b$ which is a quadratic

x	$x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
$f'(x)$	−ve	0	+ve	0	−ve
Plot $f'(x)$ in relation to the x-axis	below	on	above	on	below

x	$x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
Plotting points in relation to x-axis ⇒	below	on	above	on	below
Plot		(0, 0)		(2, 0)	

Graph of $f'(x)$



6. (a) Using synthetic division

	x^3	$+3x^2$	$-13x$	-15
$x=-1$	1	3	-13	-15
		-1	-2	15
	1	2	-15	0

Remainder = 0; $f(-1)=0$

Since the remainder = 0, -1 is a root and $(x+1)$ is a factor

$$x^3+3x^2-13x-15 = (x+1)(x^2+2x-15)$$

By factorising the second bracket further . . . $f(x)=(x+1)(x-3)(x+5)$

(b) $f(x)$ meets the x-axis when $y=0$; $(x+1)(x-3)(x+5)=0$

$(x+1)=0$ or $(x-3)=0$ or $(x+5)=0$

$x=-1, x=3, x=-5$

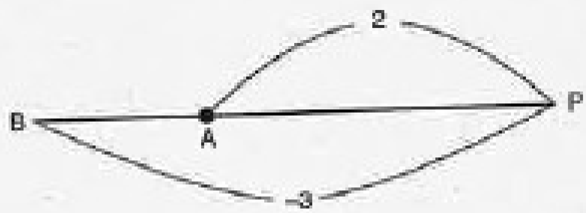
coordinates are $(-1,0)(3,0)(-5,0)$

$f(x)$ meets the y-axis when $x=0$; $f(0)=-15$; coordinate is $(0,-15)$

Hence $f(x)$ meets the axes at the points $(-5,0)$, $(-1,0)$, $(3,0)$, $(0,-15)$

7. (a) A $(2,-1,3)$, B $(1,6,-4)$

P divides AB in the ratio $2:-3$



$$\boldsymbol{a}=\begin{pmatrix}2\\-1\\3\end{pmatrix},\quad \boldsymbol{b}=\begin{pmatrix}1\\6\\-4\end{pmatrix}$$

$$\frac{\overrightarrow{AP}}{\overrightarrow{PB}}=\frac{2}{-3}$$

$$-3\overrightarrow{AP} = 2\overrightarrow{PB}$$

$$\Rightarrow\quad -3(\boldsymbol{p}-\boldsymbol{a}) = 2(\boldsymbol{b}-\boldsymbol{p})$$

$$\Rightarrow\quad -3\boldsymbol{p}+3\boldsymbol{a} = 2\boldsymbol{b}-2\boldsymbol{p}$$

$$\Rightarrow\quad 3\boldsymbol{a}-2\boldsymbol{b} = \boldsymbol{p}$$

$$\Rightarrow\quad 3\begin{pmatrix}2\\-1\\3\end{pmatrix}-2\begin{pmatrix}1\\6\\-4\end{pmatrix}=\boldsymbol{p}$$

$$\begin{pmatrix}6\\-3\\9\end{pmatrix}-\begin{pmatrix}2\\12\\-8\end{pmatrix}=\boldsymbol{p}$$

$$\begin{pmatrix}4\\-15\\17\end{pmatrix}=\boldsymbol{p}$$

$\boldsymbol{p}=\overrightarrow{OP}$ the position vector

Hence P = $(4,-15,17)$

Alternative Method:

This can also be done by section formula.

A(2, –1, 3) B(1, 6, –4)

$$P = \frac{1}{2+(-3)}\left[-3\begin{pmatrix}2\\-1\\3\end{pmatrix} + 2\begin{pmatrix}1\\6\\-4\end{pmatrix}\right]$$

$$= -1\left[\begin{pmatrix}-6\\3\\-9\end{pmatrix} + \begin{pmatrix}2\\12\\-8\end{pmatrix}\right]$$

$$= -1\begin{pmatrix}-4\\15\\-17\end{pmatrix} = (4, -15, 17), \text{ Point P}$$

(b) $\overrightarrow{AB} : \overrightarrow{PB}$

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix}1\\6\\-4\end{pmatrix} - \begin{pmatrix}2\\-1\\3\end{pmatrix} = \begin{pmatrix}-1\\7\\-7\end{pmatrix}$$

$$\overrightarrow{PB} = \boldsymbol{b} - \boldsymbol{p} = \begin{pmatrix}1\\6\\-4\end{pmatrix} - \begin{pmatrix}4\\-15\\17\end{pmatrix} = \begin{pmatrix}-3\\21\\-21\end{pmatrix} = \overrightarrow{PB}, \text{ hence } \overrightarrow{BP} = \begin{pmatrix}3\\-21\\21\end{pmatrix}$$

$\overrightarrow{AP} : \overrightarrow{PB} = 1 : 3$ and $\overrightarrow{AB} : \overrightarrow{BP} = -1 : 3$

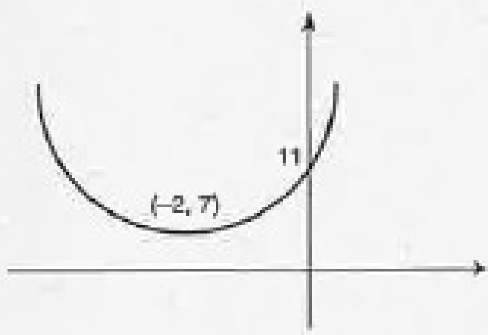
8. *(a)*

$$\begin{aligned} x^2 + 4x + 11 &= (x^2 + 4x) + 11 \\ &= (x^2 + 4x + 4) + 11 - 4 \\ &= (x + 2)^2 + 7 \end{aligned}$$

Since the least value of any square number is 0 then $(x + 2)^2$ has minimum value 0.
Hence the minimum value of the function is 0 + 7 when $x = -2$
Minimum value of the function is $(-2 + 2)^2 + 7$
and the coordinates of the minimum turning point (–2, 7).

(b) y-intersection (0, 11), minimum turning point (–2, 7).

(c) Since the minimum turning point (–2, 7) lies above the x-axis, the curve does not cross the x-axis, the equation has no real roots.



9. $u_{n+1} = 0{\cdot}4u_n + 12$;

$u_{n+1} = au_n + 12$, a limit exists when $-1 < a < 1$

a is a fraction lying between -1 and 1

Let $L = u_n$; $L = 0{\cdot}4L + 12$

$0{\cdot}6L = 12$

$L = 12 \div 0{\cdot}6$

$L = 20$

Check with $u_n = 20$; $u_{n+1} = 0{\cdot}4u_n + 12$

$u_{n+1} = 0{\cdot}4(20) + 12$; $u_{n+1} = 20$

input = output

10. $(1, 2), (a, 4), (b, 1)$

Let points = A, B, C respectively.

If points are collinear, then they have equal gradients.

i.e., $m_{AB} = m_{BC}$, B lies on line AC

$m_{AB} = \frac{4-2}{a-1}$, $m_{BC} = \frac{1-4}{b-a}$

$\Rightarrow m_{AB} = \frac{2}{a-1}$, $m_{BC} = \frac{-3}{b-a}$

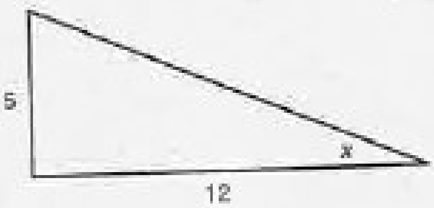
$\Rightarrow \frac{2}{a-1} = \frac{-3}{b-a}$

$\Rightarrow 2(b-a) = -3(a-1)$

$\Rightarrow 2(b-a) = -3a+3$

$\Rightarrow a + 2b = 3$

11. $\tan x = \frac{5}{12}$; $\tan y = \frac{3}{4}$



Angle x

opp = 5, adj = 12 giving hypotenuse = 13

$\sin x = \frac{5}{13}$; $\cos x = \frac{12}{13}$

Angle y

opp = 3, adj = 4 giving hypotenuse = 5

$\sin y = \frac{3}{5}$; $\cos y = \frac{4}{5}$

$\cos(x-y) = \cos x \cos y + \sin x \sin y \rightarrow \frac{12}{13}\cdot\frac{4}{5} + \frac{5}{13}\cdot\frac{3}{5} \rightarrow = \frac{63}{65}$

$\sin(x+y) = \sin x \cos y + \cos x \sin y \rightarrow \frac{5}{13}\cdot\frac{4}{5} + \frac{12}{13}\cdot\frac{3}{5} \rightarrow = \frac{56}{65}$

$\frac{63}{65} - \frac{56}{65} = \frac{7}{65}$

12. $\int_{b}^{2}(3x^2 - 2)dx = 3 \Rightarrow [x^3 - 2x]_b^2 = 3$

$$(2^3 - 2(2)) - (b^3 - 2(b)) = 3$$

$$(8 - 4) - b^3 + 2b = 3$$

$$4 - b^3 + 2b = 3$$

$$1 = b^3 - 2b$$

$$1 = b(b^2 - 2)$$

$$b = -1$$

Since $b < 2$ Trial and error

Try, $b = 1, 0$ or -1

$-1((-1)^2 - 2)$

$= -1(1 - 2) = -1(-1) = 1$