FIRRHILL HIGH SCHOOL ADVANCED HIGHER MATHEMATICS: REVISION/CATCH UP NOTES

THE BINOMIAL THEOREM: REVISION POINTS

Know how to expand $(x+y)^n$ using Pascal's Triangle.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

Know the Binomial coefficient $\binom{n}{r}$ $(= {}^nC_r)$ $= \dfrac{n!}{r!(n-r)!}$ e.g. $\binom{10}{7} = \dfrac{10\times 9\times 8}{3\times 2\times 1}$.

Know the Binomial Theorem => $(x+y)^n = \sum_{r=0}^{r=n} \binom{n}{r} x^{n-r} y^r$.

Know the General Term in $(x+y)^n$ is $T_{r+1} = \binom{n}{r} x^{n-r} y^r$ (useful to find single terms).

These revision points will be explained below with full worked examples.

Factorials and Binomial Coefficients

Factorials

Definition:

For $n \in \mathbb{N}$, the **factorial of n** (aka **n factorial** or **factorial n**) is,

$$n! \overset{def}{=} n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$$

Note that $1! = 1$ and the convention $0! \overset{def}{=} 1$ is made.

Binomial Coefficient

Definition:

The **number of ways of choosing r objects from n without taking into account the order** (aka **n choose r** or the **number of combinations of r objects from n**) is given by the **binomial coefficient** ${}^{n}C_{r}$ defined by,

$${}^{n}C_{r} \equiv \binom{n}{r} \overset{def}{=} \frac{n!}{r!\,(n - r)!}$$

Evaluating a Binomial Coefficient Without a Calculator

Example 1

$${}^{7}C_{4} = \frac{7!}{4!\,(7 - 4)!}$$

$$= \frac{7\times6\times5\times4\times3\times2\times1}{(4\times3\times2\times1)\times(3\times2\times1)}$$

$$= \frac{7\times6\times5}{3\times2\times1}$$

$$= 35$$

Properties of Binomial Coefficients

- $\binom{n}{r} = \binom{n}{n - r}$
- $\binom{n}{r - 1} + \binom{n}{r} = \binom{n + 1}{r}$

Worked Example 2

Solve the equation $\binom{n}{n-2} = 15.$

$$\frac{n!}{2!\,(n-2)!} = 15$$

$$\frac{n\,(n-1)\,(n-2)!}{2\,(n-2)!} = 15$$

$$\frac{n\,(n-1)}{2} = 15$$

$$n^2 - n = 30$$

$$n^2 - n - 30 = 0$$

$$(n-6)\,(n+5) = 0$$

Hence, $n = 6$ or $n = -5$. However, as n cannot be negative, $n = 6$.

PASCAL'S TRIANGLE

$(n = 0)$	1
$(n = 1)$	1 1
$(n = 2)$	1 2 1
$(n = 3)$	1 3 3 1
$(n = 4)$	1 4 6 4 1

This triangular array of numbers is known as **Pascal's triangle** and can be extended indefinitely.

Every row starts and ends with 1 and each number in between is the sum of the two adjacent numbers in the row above. The coefficients in every row are also symmetrical.

The Binomial Theorem

The binomial theorem helps us expand $(x + y)^n$.

Theorem (Binomial Theorem):

For whole numbers r and n,

$$(x + y)^n = \sum_{r=0}^{n} {}^nC_r \, x^{n-r} y^r$$

Written out fully, the RHS is called the **binomial expansion** of $(x + y)^n$.

Expanding a Binomial

Expand $(x + y)^5$.

$$(x + y)^5 = \sum_{r=0}^{5} \binom{5}{r} x^{5-r} y^r$$

$$= \binom{5}{0} x^5 y^0 + \binom{5}{1} x^4 y^1 + \binom{5}{2} x^3 y^2 + \binom{5}{3} x^2 y^3 + \binom{5}{4} x^1 y^4 + \binom{5}{5} x^0 y^5$$

$$(x+y)^5 = x^5 + 5x^4 y + 10x^3 y^2 + 10x^2 y^3 + 5xy^4 + y^5.$$

Note that the powers of x go up by 1 as the powers of y go down by 1, and that the sum of the powers of x and y equal 5. Also, the number of terms in the expansion is one more than the value of n . The binomial coefficients are evaluated using Pascal's triangle.

Example 1

$$\begin{aligned}(x+4)^3 &= x^3 + 3\cdot x^2 \cdot 4 + 3\cdot x \cdot 4^2 + 4^3 \\ &= x^3 + 12x^2 + 48x + 64\end{aligned}$$

Example 2

$$\begin{aligned}(y+3)^5 &= y^5 + 5\cdot y^4 \cdot 3 + 10\cdot y^3 \cdot 3^2 + 10\cdot y^2 \cdot 3^3 + 5\cdot y \cdot 3^4 + 3^5 \\ &= y^5 + 15y^4 + 90y^3 + 270y^2 + 405y + 243\end{aligned}$$

Example 3

$$(2-a)^4 = 2^4 + 4\cdot 2^3\cdot(-a) + 6\cdot 2^2\cdot(-a)^2 + 4\cdot 2\cdot(-a)^3 + (-a)^4$$
$$= 16 - 32a + 24a^2 - 8a^3 + a^4$$

Example 4

$$(x^2+3)^4 = (x^2)^4 + 4\cdot(x^2)^3\cdot 3 + 6\cdot(x^2)^2\cdot 3^2 + 4\cdot x^2\cdot 3^3 + 3^4$$
$$= x^8 + 12x^6 + 54x^4 + 108x^2 + 81$$

Example 5

$$\left(x+\frac{1}{x}\right)^5 = x^5 + 5\cdot x^4\cdot\frac{1}{x} + 10\cdot x^3\cdot\left(\frac{1}{x}\right)^2 + 10\cdot x^2\cdot\left(\frac{1}{x}\right)^3 + 5\cdot x\cdot\left(\frac{1}{x}\right)^4 + \left(\frac{1}{x}\right)^5$$
$$= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$$

Worked Example 6

Find the coefficient of x^3 in the expansion of $(2x-3)(x+2)^5$.

Solution

$$(x+2)^5 = x^5 + 5\cdot x^4\cdot 2 + 10\cdot x^3\cdot 2^2 + 10\cdot x^2\cdot 2^3 + 5\cdot x\cdot 2^4 + 2^5$$
$$= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$$

$$(2x-3)(x+2)^5 = (2x-3)(x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32)$$

Term in $x^3 = 2x\cdot 80x^2 - 3\cdot 40x^3$
$= 160x^3 - 120x^3$
$= 40x^3$

Hence the coefficient of x^3 is 40.

Finding the General Term

Definition:

The **general term** in a binomial expansion is,

$${}^nC_r\ x^{n-r}\ y^r$$

Worked Example 7

Find and simplify the general term in the expansion of $\left(x^2 + \frac{3}{x}\right)^{13}$.

$$\left(x^2 + \frac{3}{x}\right)^{13} = \sum_{r=0}^{13} \binom{13}{r} \left(x^2\right)^{13-r} \left(\frac{3}{x}\right)^r$$

The general term is the expression after the summation sign on the RHS of the above equation. So,

$$\text{General term} = \binom{13}{r}\left(x^2\right)^{13-r}\left(\frac{3}{x}\right)^r = \binom{13}{r} 3^r x^{26-2r} x^{-r} = \binom{13}{r} 3^r x^{26-3r}$$

Finding a Specific Term

Worked Example 8

Find the term independent of x in the expansion of $\left(x - \frac{5}{x^2}\right)^9$.

$$\text{General term} = \binom{9}{r} x^{9-r}\left(-\frac{5}{x^2}\right)^r = \binom{9}{r} (-5)^r x^{9-3r}$$

The term independent of x occurs when the index $9 - 3r = 0$, i.e. when $r = 3$. Thus, the required term is,

$$\binom{9}{3}(-5)^3 = \frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{(3\times2\times1)\times(6\times5\times4\times3\times2\times1)} \times (-125) = -10\,500$$

Worked Example 9

Find the term containing x^{28} in the expansion of $\left(x^2 + \frac{x}{2}\right)^{15}$.

$$\text{General term} = \binom{15}{r}\left(x^2\right)^{15-r}\left(\frac{x}{2}\right)^r = \binom{15}{r} 2^{-r} x^{30-r}$$

The term containing x^{28} occurs when the index $30 - r = 28$, i.e. when $r = 2$. Thus, the required term is,

$$\binom{15}{2} 2^{-2} x^{28} = \frac{15!}{2!\ 13!} \times \frac{1}{4} x^{28} = \frac{105}{4} x^{28}$$

Examples For You to Try….

1. Evaluate :-

(a) $\binom{5}{3}$ (b) $\binom{7}{3}$ (c) $\binom{9}{7}$

(d) $\binom{n}{2}$ (e) $\binom{n}{n-2}$ (f) $\binom{n+1}{2}$

2. Use the Binomial Theorem $(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$ or Pascal's Triangle to expand :-

(a) $(1+x)^5$ (b) $(x+y)^5$ (c) $(a-b)^3$

(d) $(1-x)^4$ (e) $(p-q)^4$ (f) $(1-4x)^3$

1. Prove that for all integers :–

(a) $\binom{n+3}{4}-\binom{n+2}{4}=\frac{n(n+1)(n+2)}{6}$, $n \geq 2$. (b) $\binom{n+2}{3}-\binom{n}{3}=n^2$, $n \geq 3$.

2. Solve the equations :–

(a) $\binom{n}{1}+\binom{n}{3}=2\binom{n}{2}$, $n > 2$. (b) $3\binom{n+2}{3}=7\binom{n+1}{2}$.

3. Use the Binomial Theorem $(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$ to expand the following :-

(a) $(2+3b)^4$ (b) $(3a+2b)^3$ (c) $(2a-3b)^3$

(d) $\left(x+\frac{1}{x}\right)^4$ (e) $\left(2x-\frac{1}{x}\right)^4$ (f) $\left(x-\frac{1}{2x}\right)^4$

4. Find the coefficients of :-

(a) x^5 in the expansion of $(1+x)^{16}$.

(b) x^4 in the expansion of $\left(1+\frac{1}{2}x\right)^{11}$.

5. By expanding or otherwise, find the :-

(a) fifth term in $\left(a-\frac{1}{3}b\right)^{12}$. (b) sixth term in $\left(2-\frac{1}{4}x\right)^{14}$.

6. Find the term independent of x in $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$.

7. Use the expansion of $(1+x)^4$ to evaluate $1\cdot03^4$ to 2 decimal places.

Solutions

1. (a) 10 (b) 35 (c) 36

   (d) $\frac{n(n-1)}{2}$ (e) $\frac{n(n-1)}{2}$ (f) $\frac{n(n+1)}{2}$

2. (a) $1+5x+10x^2+10x^3+5x^4+x^5$

   (b) $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$

   (c) $a^3-3a^2b+3ab^2-b^3$

   (d) $1-4x+6x^2-4x^3+x^4$

   (e) $p^4-4p^3q+6p^2q^2-4pq^3+q^4$

   (f) $1-12x+48x^2-64x^3$

1. (a) Proof (b) Proof

2. (a) $n=7$ (b) $n=5$

3. (a) $16+96b+216b^2+216b^3+81b^4$ (b) $27a^3+54a^2b+36ab^2+8b^3$

   (c) $8a^3-36a^2b+54ab^2-27b^3$ (d) $x^4+4x^2+6+\frac{4}{x^2}+\frac{1}{x^4}$

   (e) $16x^4-32x^2+24-\frac{8}{x^2}+\frac{1}{x^4}$ (h) $x^4-2x^2+\frac{3}{2}-\frac{1}{2x^2}+\frac{1}{16x^4}$

4. (a) 4368 (b) $\frac{165}{8}$

5. (a) $\frac{55}{9}a^8b^4$ (b) $-1001x^5$

6. $\frac{7}{18}$

7. 1·13