SCOTTISH
QUALIFICATIONS
AUTHORITY

2006 Mathematics
Advanced Higher Applied
Mechanics
Detailed Marking Instructions

Strictly Confidential

These instructions are **strictly confidential** and, in common with the scripts entrusted to you for marking, they must never form the subject of remark of any kind, except to Scottish Qualifications Authority staff. Finalised Marking Instructions will be published on SQA's website in due course.

Markers' Meeting

You should use the time before the meeting to make yourself familiar with these instructions and any scripts which you have received. Do **not** undertake any final approach to marking until **after** the meeting. Please note any points of difficulty for discussion at the meeting.

Note: These instructions can be considered as final only after the markers' meeting when the full marking team has had an opportunity to discuss and finalise the document in the light of a wider range of candidates' responses.

Marking

The utmost care must be taken when entering and totalling grades. Where appropriate, all summations for totals must be carefully checked and confirmed.

Recording of Grades

Always enter the **Total** grade (using red ink) as a **whole number**, where necessary by the process of rounding up.

All entries on the Mark Sheet, including the grade assigned, must be made in red.

Markers are reminded that they must not write comments on scripts.

22 May 2006

Section A – Mechanics

A1.

$$\mathbf{r}(t) = \left(\frac{1}{3}t^3 - 4t^2\right)\mathbf{i} - \left(2t^2 - 1\right)\mathbf{j}$$

$$\Rightarrow \mathbf{v}(t) = \left(t^2 - 8t\right)\mathbf{i} - 4t\mathbf{j} \qquad 1$$

$$\Rightarrow \mathbf{a}(t) = (2t - 8)\mathbf{i} - 4\mathbf{j} \qquad 1$$

If **a** is in the **j** direction then $t = 4$ so 1

$$\mathbf{v}(4) = -16\mathbf{i} - 16\mathbf{j} \qquad 1$$

$$|\mathbf{v}(4)| = \sqrt{16^2 + 16^2} = 16\sqrt{2} \qquad 1$$

A2. When $v = v_{\max}$, $x = 0$, so

$$v_{\max} = \omega a = \frac{1}{4}\omega \qquad 1$$

Using

$$v^2 = \omega^2\left(a^2 - x^2\right) \qquad \text{with} \qquad a = \frac{1}{4}$$

$$\text{gives} \qquad v^2 = \omega^2\left(\frac{1}{16} - x^2\right). \qquad 1$$

But $v = \frac{1}{2}v_{\max} = \frac{1}{8}\omega$ so

$$\frac{\omega^2}{64} = \omega^2\left(\frac{1}{16} - x^2\right) \qquad 1$$

$$x^2 = \frac{1}{16} - \frac{1}{64} = \frac{3}{64} \qquad 1$$

$$x = \frac{\pm\sqrt{3}}{8}$$

i.e. the distance from O is $\frac{\sqrt{3}}{8}$ metres. 1

A3. (a) $\mathbf{a}_L = \frac{1}{8}g\mathbf{j}$, $\mathbf{v}_L(0) = \mathbf{0}$, $\mathbf{r}_L(0) = \mathbf{0}$

$$\Rightarrow \mathbf{v}_L(t) = \frac{gt}{8}\mathbf{j}$$

$$\Rightarrow \mathbf{r}_L(t) = \frac{gt^2}{16}\mathbf{j} \qquad 1$$

Also

$$\mathbf{a}_B = -g\mathbf{j}, \qquad \mathbf{v}_B(0) = \mathbf{0}, \qquad \mathbf{r}_B(0) = 2\mathbf{j}$$

$$\Rightarrow \mathbf{v}_B(t) = -gt\mathbf{j}$$

$$\Rightarrow \mathbf{r}_B(t) = \left(2 - \frac{1}{2}gt^2\right)\mathbf{j} \qquad 1$$

$${}_B\mathbf{r}_L = \left[\frac{-gt^2}{16} + 2 - \frac{gt^2}{2}\right]\mathbf{j} \qquad 1$$

$$= \left[\frac{-9gt^2}{16} + 2\right]\mathbf{j}$$

(b) When ${}_B\mathbf{r}_L = \mathbf{0}$, $gt^2 = \frac{32}{9}$ so **1**

$$\mathbf{r}_B = \left(2 - \frac{1}{2} \times \frac{32}{9}\right)\mathbf{j} = \frac{2}{9}\mathbf{j}$$ **1**

Distance light bulb falls = $2 - \frac{2}{9} = \frac{16}{9}$ metres. **1**

A4. (a) Using $\ddot{x} = 0, \ddot{y} = -g, \mathbf{v} = V\cos\alpha\mathbf{i} + V\sin\alpha\mathbf{j}$

$$x = V\cos\alpha\, t \qquad y = V\sin\alpha\, t - \frac{1}{2}gt^2$$ **1,1**

When $y = 0$, $\quad t = \frac{2V}{g}\sin\alpha \quad (t > 0)$ **1**

$$\text{Range,}\quad R = V\cos\alpha \times \frac{2V}{g}\sin\alpha$$

$$= \frac{V^2}{g}(2\sin\alpha\cos\alpha)$$ **1**

$$= \frac{V^2\sin 2\alpha}{g}.$$

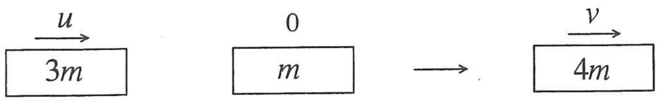
(b) With $\alpha = 15°$ and $R > L$ and $R < 2L$, **1**

$$R > L \quad\Rightarrow\quad \frac{V^2}{gL}\sin 30° > 1 \quad\Rightarrow \frac{V^2}{gL} > 2 \quad\Rightarrow\quad \frac{V}{\sqrt{gL}} > \sqrt{2}$$ **1**

$$R < 2L \quad\Rightarrow\quad \frac{V^2}{2gL} < 2 \quad\Rightarrow\quad \frac{V^2}{gL} < 4 \quad\Rightarrow\quad \frac{V}{\sqrt{gL}} < 2$$ **1**

$$\text{i.e.}\quad \sqrt{2} < \frac{V}{\sqrt{gL}} < 2.$$

A5. (a)



$$P_{\text{before}} = 3mu; \qquad P_{\text{after}} = 4mv$$

By conservation of momentum

$$v = \tfrac{3}{4}u$$ **1**

Using $v = u + at, 0 = \frac{3}{4}u + aT \Rightarrow a = \frac{-3u}{4T}$. **1**

and $-R = -4m \times \frac{3u}{4T}$ **1**

$$\Rightarrow R = \frac{3mu}{T}$$ **1**

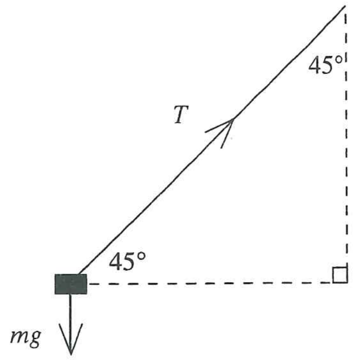
(b) Using $s = ut + \frac{1}{2}at^2$ gives

$$\text{Distance to rest} = \frac{3}{4}uT - \frac{1}{2}\frac{3u}{4T}T^2$$ **1**

$$= \frac{3}{4}uT\left(1 - \frac{1}{2}\right) = \frac{3uT}{8}$$ **1**

$$\text{Work done} = \frac{3uT}{8} \times \frac{3mu}{T} = \frac{9}{8}mu^2 \text{ Nm}$$ **1**

A6.



$$T = \frac{\lambda x}{l} = \frac{8mgx}{l}$$ **1**

(a) Resolving forces vertically

$$\frac{\lambda x}{l}\cos 45° = mg$$ **1**

$$\frac{8mgx}{l\sqrt{2}} = mg$$

$$x = \frac{l}{4\sqrt{2}} = \frac{l\sqrt{2}}{8}$$ **1**

(b) The length of the string is $x + l$.
Resolving horizontally

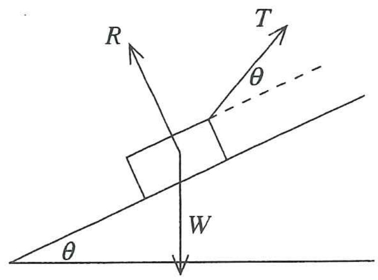
$$m(x + l)\cos 45°\omega^2 = T\sin 45°$$ **1**

$$\frac{m}{\sqrt{2}}\left(1 + \frac{1}{4\sqrt{2}}\right)l\omega^2 = \frac{8mg}{l} \times \frac{l}{4\sqrt{2}} \times \frac{1}{\sqrt{2}}$$ **1**

$$\frac{4\sqrt{2} + 1}{8}l\omega^2 = \frac{8g}{8}$$ **1**

$$\therefore \quad \omega^2 = \frac{8g}{(1 + 4\sqrt{2})\,l}$$

A7. (a)



Resolving perpendicular to the plane

$$W\cos\theta = R + T\sin\theta \qquad (1)\ \mathbf{1}$$

Resolving parallel to the plane

$$W\sin\theta + \mu R = T\cos\theta \qquad (2)\ \mathbf{1}$$

From (1)

$$R = W\cos\theta - T\sin\theta.$$

Substituting into (2)

$$W\sin\theta + \mu(W\cos\theta - T\sin\theta) = T\cos\theta \qquad \mathbf{1}$$

$$\Rightarrow \quad T(\cos\theta + \mu\sin\theta) = (\sin\theta + \mu\cos\theta)W \qquad \mathbf{1}$$

$$\Rightarrow \quad T = \frac{\sin\theta + \mu\cos\theta}{\cos\theta + \mu\sin\theta}W \qquad \mathbf{1}$$

$$\Rightarrow \quad T = \frac{\frac{\sin\theta}{\cos\theta} + \mu}{1 + \mu\frac{\sin\theta}{\cos\theta}}W \qquad \mathbf{1}$$

$$\Rightarrow \quad T = \frac{\tan\theta + \mu}{1 + \mu\tan\theta}W.$$

(b) We require $T < W$

$$\text{i.e.}\left(\frac{\tan\theta + \mu}{1 + \mu\tan\theta}\right)W < W \qquad \mathbf{1}$$

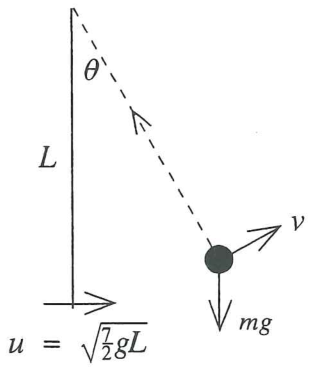
$$\tan\theta + \mu < 1 + \mu\tan\theta \qquad \mathbf{1}$$

$$\tan\theta(1 - \mu) < 1 - \mu$$

$$\tan\theta < 1 \text{ since } 0 < \mu < 1 \text{ and } 0 < \theta < \frac{\pi}{2} \qquad \mathbf{1}$$

$$\text{Thus} \quad 0 < \theta < \frac{\pi}{4}. \qquad \mathbf{1}$$

A8.



(a) (i) By energy conservation **1**

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgL(1 - \cos\theta) \quad \mathbf{1}$$

$$\frac{1}{2}\cdot\frac{7}{2}gL = \frac{1}{2}v^2 + gL(1 - \cos\theta)$$

$$\text{so } v^2 = \left(\frac{3}{2} + 2\cos\theta\right)gL \qquad (1) \quad \mathbf{1}$$

When $\theta = 45°$

$$v^2 = \frac{(3 + 2\sqrt{2})gL}{2}$$

$$v = \sqrt{\frac{(3 + 2\sqrt{2})gL}{2}} \quad \mathbf{1}$$

(ii) Resolving forces along the line of the string

$$T = \frac{mv^2}{L} + mg\cos\theta \quad \mathbf{1}$$

With $\theta = 45°$

$$T = \frac{m}{L}\frac{(3 + 2\sqrt{2})gL}{2} + \frac{mg}{\sqrt{2}} \quad \mathbf{1}$$

$$= \frac{mg}{2}\{3 + 2\sqrt{2} + \sqrt{2}\}$$

$$= \frac{3}{2}(1 + \sqrt{2})mg \quad \mathbf{1}$$

(b) When the string goes slack, $T = 0$ so

$$v^2 = -gL\cos\theta, \quad \mathbf{1}$$

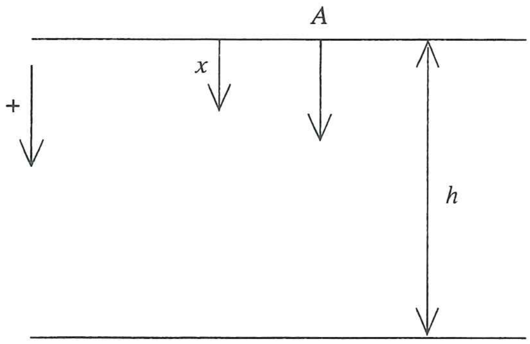
so, in (1)

$$-gL\cos\theta = \left(\frac{3}{2} + 2\cos\theta\right)gL \quad \mathbf{1}$$

$$\Rightarrow \quad 3\cos\theta = -\frac{3}{2} \quad \mathbf{1}$$

$$\Rightarrow \quad \cos\theta = -\frac{1}{2} \quad \Rightarrow \quad \theta = 120° \quad \mathbf{1}$$

A9.



(a)

$$ma = mg - mkv^2 \qquad \mathbf{1}$$

$$v\frac{dv}{dx} = g - kv^2 \qquad \mathbf{1}$$

(b)

$$\int \frac{v\,dv}{g - kv^2} = \int dx \qquad \mathbf{1}$$

Let $\quad w = g - kv^2$

$$dw = -2kv\,dv$$

$$\frac{-1}{2k}\int \frac{dw}{w} = x + c \qquad \mathbf{1}$$

$$\frac{-1}{2k}\ln\left|g - kv^2\right| = x + c \qquad \mathbf{1}$$

As $v = 0$ when $x = 0$, $c = \dfrac{-1}{2k}\ln g$ $\qquad \mathbf{1}$

$$x = \frac{1}{2k}\ln g - \frac{1}{2k}\ln\left|g - kv^2\right|$$

$$= \frac{1}{2k}\ln\left|\frac{g}{g - kv^2}\right| \qquad \mathbf{1}$$

$$2kx = \ln\left|\frac{g}{g - kv^2}\right|$$

$$e^{2kx} = \frac{g}{g - kv^2}$$

$$ge^{2kx} - kv^2e^{2kx} = g \qquad \mathbf{1}$$

$$kv^2e^{2kx} = g\left(e^{2kx} - 1\right)$$

$$v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right)$$

(c) When $x = h$ final KE $= \frac{1}{2}mv^2 = \frac{mg}{2k}\left(1 - e^{-2kh}\right) = \frac{mg}{2k}\left(1 - e^{-4}\right)$ $\qquad \mathbf{1}$

Work done against resistance as fraction of initial PE

$$= \frac{mgh - \frac{mg}{2k}\left(1 - e^{-4}\right)}{mgh} \qquad \mathbf{1}$$

$$= \frac{1 - \frac{1}{4}\left(1 - e^{-4}\right)}{1} \approx 0.755 \qquad \mathbf{1}$$

Section B – Mathematics

B1.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right)$$ **M1, 2E1**

$$\text{So } A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & 1 \end{pmatrix}.$$

Other valid methods will be accepted.

$$\begin{array}{ccccccc} x & + & y & & & = & 1 \\ 2x & + & 3y & + & z & = & 2 \\ 2x & + & 2y & + & z & = & 1 \end{array}$$

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$ **M1,1**

so $x = 0, y = 1, z = -1$.

B2.

$$y = \ln(1 + \sin x)$$

$$\frac{dy}{dx} = \frac{\cos x}{1 + \sin x}$$ **M1,1**

$$\text{so } \frac{d^2y}{dx^2} = \frac{(1 + \sin x)(-\sin x) - \cos x \cos x}{(1 + \sin x)^2}$$ **M1,1**

$$= \frac{-\sin x - 1}{(1 + \sin x)^2}$$ **1**

$$= \frac{-1}{(1 + \sin x)}.$$

B3.

$$S_n = \tfrac{1}{6}n(n + 1)(2n + 1)$$ **1**

$$S_{2n+1} = \tfrac{1}{6}(2n + 1)(2n + 2)(4n + 3)$$ **1**

$$2^2 + 4^2 + \ldots + (2n)^2 = 4(1^2 + 2^2 + \ldots + n^2)$$

$$= \tfrac{2}{3}n(n + 1)(2n + 1)$$ **1**

B4.

$$\cos^2 y \frac{dy}{dx} = y$$

$$\int \frac{dy}{y} = \int \sec^2 x \, dx \qquad \textbf{M1}$$

so $\ln y = \tan x + c.$ **1,1**

When $y = 2, x = 0$ giving $c = \ln 2.$ **1**

Hence $\ln y - \ln 2 = \tan x$, i.e. $\ln \tfrac{1}{2}y = \tan x$

$$\Rightarrow y = 2e^{\tan x}. \qquad \textbf{1}$$

B5. $1 + x^2 = u \Rightarrow x\,dx = \tfrac{1}{2}\,du$ so **1**

$$\int \frac{x^3}{\sqrt{1 + x^2}}\,dx = \int \frac{(u - 1)}{\sqrt{u}} \frac{1}{2}\,du \qquad \textbf{1}$$

$$= \frac{1}{2} \int (u^{1/2} - u^{-1/2})\,du \qquad \textbf{1}$$

$$= \tfrac{1}{3}u^{3/2} - u^{1/2} + c \qquad \textbf{1}$$

$$= \tfrac{1}{3}(1 + x^2)^{3/2} - (1 + x^2)^{1/2} + c \qquad \textbf{1}$$

$$= \tfrac{1}{3}(x^2 - 2)\sqrt{1 + x^2} + c$$

B6. (a)

$$\int_0^1 x\,e^{2x}\,dx = \left[x \int e^{2x}\,dx - \int \tfrac{1}{2}\,e^{2x}\,dx\right]_0^1 \qquad \textbf{M1, 1}$$

$$= \left[\tfrac{1}{2}x\,e^{2x} - \tfrac{1}{4}\,e^{2x}\right]_0^1 \qquad \textbf{1}$$

$$= \tfrac{1}{2}e^2 - \tfrac{1}{4}e^2 + \tfrac{1}{4} = \tfrac{1}{4}(e^2 + 1) \qquad \textbf{1}$$

(b)

$$\int_0^1 x^2\,e^{2x}\,dx = \left[x^2 \int e^{2x}\,dx\right]_0^1 - \int_0^1 2x.\tfrac{1}{2}\,e^{2x}\,dx \qquad \textbf{1}$$

$$= \left[\tfrac{1}{2}x^2e^{2x}\right]_0^1 - \int_0^1 x\,e^x\,dx \qquad \textbf{1}$$

$$= \left[\tfrac{1}{2}e^2 - 0\right] - \tfrac{1}{4}(e^2 + 1) = \tfrac{1}{4}(e^2 - 1) \qquad \textbf{1}$$

(c)

$$\int_0^1 (3x^2 + 2x)\,e^{2x}\,dx = 3\int_0^1 x^2\,e^{2x}\,dx + 2\int_0^1 x\,e^{2x}\,dx \qquad \textbf{1}$$

$$= \tfrac{3}{4}(e^2 - 1) + \tfrac{2}{4}(e^2 + 1) \qquad \textbf{1}$$

$$= \tfrac{1}{4}(5e^2 - 1)$$

[END OF MARKING INSTRUCTIONS]