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2003 Applied Mathematics

Advanced Higher

Finalised Marking Instructions

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2003 Applied Mathematics

Advanced Higher – Section A

Finalised Marking Instructions

Advanced Higher Applied 2003: Section A Solutions and marks

A1. P(Taxi Yellow | Witness states Yellow)

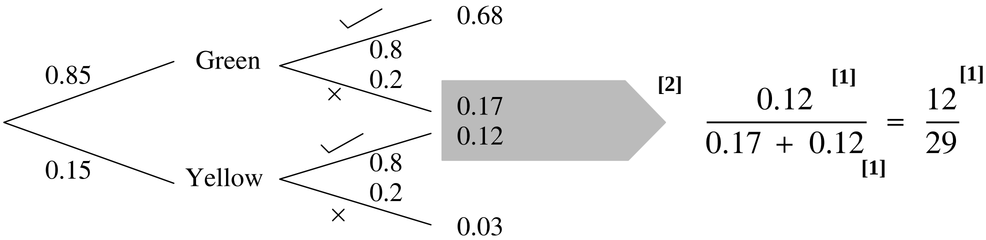
$= \dfrac{\text{P(Taxi Yellow} \cap \text{Witness states Yellow)}}{\text{P(Witness states Yellow)}}$ **M1**

$= \dfrac{\text{P(Witness states Yellow | Taxi Yellow).P(Taxi Yellow)}}{\text{P(WY | TY).P(TY) + P(WY | TG).P(TG)}}$ **M1**

$= \dfrac{0.8 \times 0.15}{0.8 \times 0.15 + 0.2 \times 0.85}$ **1, 1**

$= \dfrac{0.12}{0.29} = 0.41$ **1**

Alternative



A2. T_1 has mean 180 and s.d. 12
T_2 has mean 20 and s.d. 5

Total $T = T_1 + T_2$ has mean 180 + 20 = 200 **1**
variance $12^2 + 5^2 = 169$ **1**
so s.d. $= 13$ **1**

Assuming that T is normally distributed **1**
the required value of z is 3.09 **1**

$T = \mu + z\sigma = 200 + 3.09 \times 13 = 240.17$
$x = 241$ **1**

A3. (a) Assemble the lists into a single list of pupils numbered 1 to 180. **1**

Required sample size is 10% of 180 = 18.

Select a number at random from 1 to 10 e.g. 3 **1**

Take the 3rd pupil from the list and every 10th pupil thereafter. **1**

(b) Possible advantages: (just one needed) **1**

- systematic sampling is easily implemented
- helps to provide a good spread

Possible disadvantages: (just one needed) **1**

- if there is a pattern in the population matched by the sequence an unrepresentative sample will result
- in order to number the population, its size has to be known.

A4. $X \sim Bin(100,\ 0.75)$ **1**

$\Rightarrow X$ is approximately N **1**

and $N(75,\ 4.33^2)$ **1**

$P(X \leqslant 70) = P\left(Z \leqslant \frac{70.5 - 75}{4.33}\right)$ **1**

$= P(Z \leqslant -1.04)$ **1**

$= 0.1492$ **1**

A5. (a) $t = \frac{r}{\sqrt{\frac{1-r^2}{n-2}}} = \frac{-0.655}{\sqrt{\frac{1-(-0.655)^2}{59}}}$

$= -6.66.$ **1**

The critical region for t at the 5% level of significance with 59 d.f. **1**

will be approx. $|t| > 1.96$. **1**

Since -6.66 lies in the critical region H_0 would be rejected **1**

(b) there is evidence of a linear relationship between x and y **1**

A6. $P(X < 10 + a) = F(9 + a)$ where $F(.)$ is the Poi(1) distribution function. **1**

$P(10 - a < X < 10 + a) = F(9 + a) - F(10 - a)$ **1**

We require $F(9 + a) - F(10 - a) \geqslant 0.99$

$a = 7$ gives $F(16) - F(3) = 0.9626$

$a = 8$ gives $F(17) - F(2) = 0.9830$ **M1**

$a = 9$ gives $F(18) - F(1) = 0.9923$

Smallest integer is $a = 9$. [Method has to be clear.] **1**

A7. (a) $H_0 : \mu_D = 0$

$H_1 : \mu_D > 0$ [Must infer *differences*] **1**

$t = \frac{\bar{d} - \mu_D}{\frac{s_D}{\sqrt{n}}} = \frac{3.83 - 0}{\frac{5.41}{\sqrt{12}}}$

$= 2.45$ **1**

The critical region at the 5% signficance level with 11 df is $t > 1.796$. **1**

Thus the null hypothesis would be rejected

at the 5% level of significance **1**

so the data do provide evidence that the training course has been effective. **1**

(b) A sign test could have been used. **1**

A8. (a) Assume that the journey time is normally distributed (with $\sigma = 3$). **1**

$H_0 : \mu = 28$

$H_1 : \mu \neq 28$ [Must be two-tailed] **1**

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{25.125 - 28}{\frac{3}{\sqrt{8}}}$$

$= -2.71$ **1**

The critical region is $z < -2.58$ or $z > 2.58$. **1**

Since $-2.71 < -2.58$ the null hypothesis would be rejected **1**

at the 1% level of significance i.e. there is evidence of a change. **1**

(b) p-value = $2 \times \Phi(-2.71)$ **1**

$= 2(1 - 0.9966) = 0.0068$ **1**

The fact that the p-value is less than 0.01 confirms rejection of the null hypothesis at the 1% level of significance **1**

(c) The fact that 28 does not lie in the 99% confidence interval **1**

confirms rejection of the null hypothesis at the 1% level.

A9. (a) $p = 78/100 = 0.78$.

A 95% C.I. for the proportion of ischaemic strokes in the population is

$$0.78 \pm 1.96\sqrt{\frac{0.78 \times 0.22}{100}}$$ **1**

$$0.78 \pm 0.08$$ **1**

or (0.70, 0.86)

(b) The interval does not include 0.65 which means that **1**

there is evidence of differing proportions. **1**

(c) Observed

	Died	Survived	
Isch.	37	41	78
Haem.	15	7	22
	52	48	100

Expected

	Died	Survived	
Isch.	40.56	37.44	78
Haem.	11.44	10.56	22
	52	48	100

1,1

H_0 : Survival is independent of the type of stroke.

H_1 : Survival is dependent of the type of stroke. **1**

$$X^2 = \sum \frac{(O - E)^2}{E}$$

$x^2 = 0.312 + 0.339 + 1.108 + 1.200$

$= 2.959$. **1**

Since $\chi^2_{5\%, 1\ df} = 3.841 > 2{\cdot}959$ **1**

H_0 is accepted at the 5% level **1**

i.e. there is no evidence that survival depends on the type of stroke. **1**

A10. (a) Since all the sample means plot within the chart limits there is no evidence of special cause variation. **1**

$\mu = 5018.86$ **1**

Limits are given by $\mu \pm 3\dfrac{\sigma}{\sqrt{n}}$ **1**

$5018.86 \pm 3\dfrac{288.3}{\sqrt{5}}$ [= 5018.86 ± 386.79 ~ 4632.1, 5405.6 as on chart]. **1**

(b)

$P(4500 < \text{Volume} < 5500)$ **1**

$= \Phi\left(\dfrac{5500 - 5018.86}{288.3}\right) - \Phi\left(\dfrac{4500 - 5018.86}{288.3}\right)$ **1**

$= \Phi(1.67) - \Phi(-1.80)$ **1**

$= 0.9176$ **1**

(c) Adjust process so that mean becomes 5000. **1**

Reduce the variability in the process. **1**

[END OF MARKING INSTRUCTIONS]

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2003 Applied Mathematics

Advanced Higher – Section B

Finalised Marking Instructions

Advanced Higher Applied 2003: Section B Solutions and marks

B1. $f(x) = \sqrt{9-4x}$, $\quad f'(x) = \dfrac{-2}{(9-4x)^{1/2}} \quad f''(x) = \dfrac{-4}{(9-4x)^{3/2}} \quad f'''(x) = \dfrac{-24}{(9-4x)^{5/2}}$

Taylor polynomial is

$$p(2+h) = 1 - 2h - \frac{4h^2}{2} - \frac{24h^3}{6}$$

$$= 1 - 2h - 2h^2 - 4h^3. \qquad \mathbf{3}$$

Second degree approximation is $p(2 + 0{\cdot}03) = 1 - 0{\cdot}06 - 0{\cdot}0018 = 0{\cdot}9382$ **2**

Principal truncation error term is $-4 \times 0{\cdot}03^3 = -0{\cdot}0001$.
Hence second order estimate cannot be guaranteed accurate to 4 decimal places. **2**

B2.

$$L(x) = \frac{(x-0{\cdot}2)(x-0{\cdot}5)}{(-0{\cdot}2)(-0{\cdot}5)}1{\cdot}306 + \frac{(x-0{\cdot}0)(x-0{\cdot}5)}{(0{\cdot}2)(-0{\cdot}3)}1{\cdot}102 + \frac{(x-0{\cdot}0)(x-0{\cdot}2)}{(0{\cdot}5)(0{\cdot}3)}0{\cdot}741$$

$$= (x^2 - 0{\cdot}7x + 0{\cdot}1)13{\cdot}06 - (x^2 - 0{\cdot}5x)18{\cdot}367 + (x^2 - 0{\cdot}2x)4{\cdot}490$$

$$= -0{\cdot}367x^2 - 0{\cdot}947x + 1{\cdot}306 \qquad \mathbf{4}$$

B3. The first relation is linear since there is no term in a_r of more than first degree. **1**
Relation (i) is a second order relation. Its fixed point a is found from
$2a = 3a - 4a + 9$, i.e. $a = 3$. **2**
Sequence from (ii) is $a_0 = 1;\ a_1 = 1;\ a_2 = \frac{1}{2};\ a_3 = -3/8$. **2**

B4. Let quadratic through (x_0, f_0), (x_1, f_1), (x_2, f_2) be

$$y = A_0 + A_1(x - x_0) + A_2(x - x_0)(x - x_1).$$

Then $f_0 = A_0;\ f_1 = A_0 + A_1h;\ f_2 = A_0 + 2A_1h + 2A_2h^2$
and so

$$A_1 = \frac{f_1 - f_0}{h} = \frac{\Delta f_0}{h}; \qquad A_2 = \frac{f_2 - 2f_1 + f_0}{2h^2} = \frac{\Delta^2 f_0}{2h^2}.$$

Thus

$$y = f_0 + \frac{x - x_0}{h}\Delta f_0 + \frac{(x - x_0)(x - x_1)}{2h^2}\Delta^2 f_0.$$

Setting $x = x_0 + ph$ gives

$$y = f_0 + p\Delta f_0 + \tfrac{1}{2}p(p-1)\Delta^2 f_0. \qquad \mathbf{5}$$

(Can also be done by an operator expansion of $(1 + \Delta)^p$.)

B5.
(a) Maximum error is 8ε, i.e. $8 \times 0{\cdot}0005 = 0{\cdot}004$. **1**
(b) $\Delta^2 f_3 = 0{\cdot}167$. **1**
(c) Third degree polynomial would probably not be particularly good as an approximation as differences are not constant. **1**
(d) Working from $x = 2{\cdot}0$, $p = 0{\cdot}9$.

$$f(2{\cdot}18) = 2{\cdot}318 + 0{\cdot}9(0{\cdot}197) + \frac{(0{\cdot}9)(-0{\cdot}1)}{2}(0{\cdot}086)$$

$$= 2{\cdot}318 + 0{\cdot}177 - 0{\cdot}004 = 2{\cdot}491 \qquad \mathbf{2}$$

B6. Gauss-Seidel table is:

x_1	x_2	x_3
0	0	0
1·625	−3·642	−0·348
2·014	−3·616	−0·347
2·011	−3·616	−0·347

Hence, to 2 decimal places, $x_1 = 2{\cdot}01$; $x_2 = -3{\cdot}62$; $x_3 = -0{\cdot}35$. **5**

B7. For $g(x) = (1 + x^5)/3$, $g'(x) = 5x^4/3$.
In $I_2 = [1{\cdot}22, 1{\cdot}3]$, $g'(x) > 1$, so clearly unsuitable.
In $I_1 = [0{\cdot}0, 0{\cdot}5]$, $g'(x) << 1$, so probably suitable. **2**

Simple Iteration gives $x_0 = 0$, $x_1 = 0{\cdot}33333$, $x_2 = 0{\cdot}33471$, $x_3 = 0{\cdot}33473$
so that root is 0·3347 (to 4 decimal places). **2**

For bisection, $f(1{\cdot}2) = -0{\cdot}112$; $f(1{\cdot}3) = 0{\cdot}813$
$f(1{\cdot}25) = 0{\cdot}302$
$f(1{\cdot}225) = 0{\cdot}084$;
$f(1{\cdot}2125) = -0{\cdot}017$
Hence root lies in [1·2125, 1·225]. **3**

B8. (a) Simpson's rule calculation is:

x	$f(x)$	m_1	$m_1 f(x)$	m_2	$m_2 f(x)$
0	0·0	1	0·0	1	0·0
0·25	0·04868			4	0·19472
0·5	0·15163	4	0·60653	2	0·30326
0·75	0·26571			4	1·06284
1	0·36788	1	0·36788	1	0·36788
			0·97441		1·92870

Hence $I_2 = 0{\cdot}97441 \times 0{\cdot}5/3 = 0{\cdot}16240$
and $I_4 = 1{\cdot}92870 \times 0{\cdot}25/3 = 0{\cdot}16072$ **4**

(b) $f^{\text{iv}}(0) = 12$; $f^{\text{iv}}(1) = 1{\cdot}84$.
Maximum truncation error $\approx 12 \times 0{\cdot}25^4/180 = 0{\cdot}00026$. **2**
Hence suitable estimate is $I_4 = 0{\cdot}161$. **1**

(c) With n strips and step size $2h$, the Taylor series for expansion of an integral approximated by Simpson's rule (with principal truncation error of $O(h^4)$) is
$I = I_n + C(2h)^4 + D(2h)^6 + \ldots = I_n + 16Ch^4 + \ldots$ (1)
With $2n$ strips and step size h, we have
$I = I_{2n} + Ch^4 + Dh^6 + \ldots$ (2)
$16 \times (2) - (1)$ gives $15I = 16I_{2n} - I_n + O(h^6)$
i.e. $I \approx (16I_{2n} - I_n)/15 = I_{2n} + (I_{2n} - I_n)/15$ **3**
$I = 0{\cdot}16072 + (0{\cdot}16072 - 0{\cdot}16240)/15 = 0{\cdot}16061$
or 0·1606 to suitable accuracy. **1**

B9. Gaussian elimination table is:

					sum
	(4·1)	−5·7	1·4	4·9	4·7
	1·6	−2·2	0·5	2·2	2·1
	0	(1·5)	2·2	−0·8	2·9
$R_2 - 1{\cdot}6R_1/4{\cdot}1$	0	0·024	−0·046	0·288	0·266
$R_4 - 0{\cdot}024R_3/1{\cdot}5$	0	0	−0·081	0·301	0·220

$x_3 = -3{\cdot}716$ $\quad x_2 = 4{\cdot}917$ $\quad x_1 = 9{\cdot}300$

To 1 decimal place:

$x_1 = 9{\cdot}3$ $\quad x_2 = 4{\cdot}9$ $\quad x_3 = -3{\cdot}7$

6

Ill-conditioning means that a small change in the element(s) of the input data is likely to cause a large change in the solution of the equations.

The fact that the new elements in rows 4 and 5 of the tableau are much smaller than the original elements (or small size of determinant) suggests ill-conditioning. **3**

B10. Euler tables are:

x	y	y'	x	y	y'
1	1	0·2679	1	1	0·2679
1·1	1·0268	0·2036	1·05	1·0134	0·2355
1·2	1·0472		1·1	1·0252	0·2041
			1·15	1·0354	0·1737
			1·2	1·0441	
($h = 0{\cdot}1$)			($h = 0{\cdot}05$)		

Hence $y_A(1{\cdot}2) = 1{\cdot}0472$ $(h = 0.1)$

$y_B(1{\cdot}2) = 1{\cdot}0441$ $(h = 0{\cdot}05)$. **4**

Truncation error is linear (first order) in h. **1**

Let E = magnitude of truncation error in $y_B(1{\cdot}2)$. Then, since error is linear in h,

$$y_A(1{\cdot}2) - 2E = y_B(1{\cdot}2) - E$$

i.e. $1{\cdot}0472 - 2E = 1{\cdot}0441 - E \Rightarrow E \approx 0{\cdot}0031$ (or $0{\cdot}003$).

Hence $y(1{\cdot}2) = 1{\cdot}0410$, rounded to suitable accuracy as 1·04. **2**

Estimate of error is probably too large (maximum truncation error). **1**

Rounding error is not likely to be important as the calculation is performed to 4 decimal places and the answer given to 2 decimal places. **2**

[END OF MARKING INSTRUCTIONS]

2003 Applied Mathematics

Advanced Higher – Section C

Finalised Marking Instructions

Advanced Higher Applied 2003: Section C Solutions and marks

C1. (a) We are given that $\frac{d^2x}{dt^2} = 12 - 3t^2, v(0) = 0, s(0) = 0$

$$\Rightarrow \quad v(t) = 12t - t^3 \qquad \mathbf{1}$$

$$\Rightarrow \quad s(t) = 6t^2 - \frac{1}{4}t^4. \qquad \mathbf{1}$$

When the particle is at rest

$$v(t) = 0 \quad \Rightarrow \quad t = 0 \text{ or } t^2 = 12$$

$$\Rightarrow \quad t = 2\sqrt{3} \text{ seconds} \qquad \mathbf{1}$$

The distance from the origin at this time is

$$s(2\sqrt{3}) = 12\left(6 - \tfrac{1}{4} \times 12\right) = 36 \text{ m} \qquad \mathbf{1}$$

(b) When the particle returns to the origin

$$s(t) = 0 \quad \Rightarrow \quad t^2\left(6 - \tfrac{1}{4}t^2\right) = 0 \qquad \mathbf{1}$$

$$\Rightarrow \quad t^2 = 24 \text{ or } t^2 = 0$$

$$\Rightarrow \quad t = 2\sqrt{6} \text{ (since } t > 0). \qquad \mathbf{1}$$

The velocity at this time is

$$\mathbf{v} = 2\sqrt{6}(12 - 24)\mathbf{i} = -24\sqrt{6}\mathbf{i} \text{ ms}^{-1} \qquad \mathbf{1}$$

C2. (a) Given $\mathbf{a}_A = -2\mathbf{j}; \mathbf{v}_A(0) = \mathbf{i}; \mathbf{r}_A(0) = -\mathbf{i}$

$$\mathbf{v}_A(t) = -2t\mathbf{j} + \mathbf{c} = \mathbf{i} - 2t\mathbf{j} \qquad \mathbf{1}$$

$$\Rightarrow \quad \mathbf{r}_A(t) = t\mathbf{i} - t^2\mathbf{j} - \mathbf{i} = (t - 1)\mathbf{i} - t^2\mathbf{j} \qquad \mathbf{1}$$

(b) (i)

$$_A\mathbf{r}_B = \mathbf{r}_A - \mathbf{r}_B = (2 - t)\mathbf{i} - \mathbf{j} \qquad \mathbf{1}$$

(ii) The square of the distance between A and B is

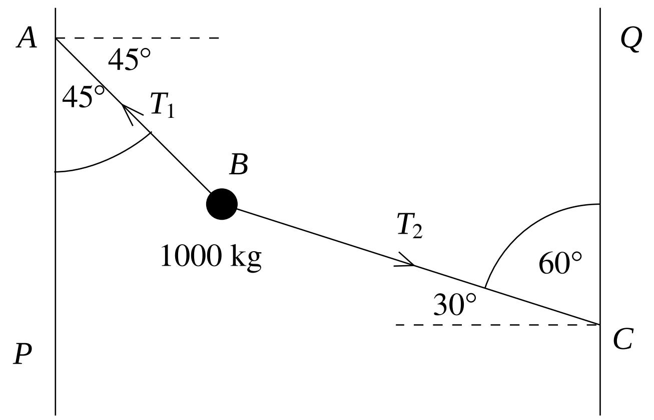
$$|_A\mathbf{r}_B|^2 = (2 - t)^2 + 1. \qquad \mathbf{1}$$

This has minimum when $t = 2$, **1**

and the minimum distance is 1 metre. **1**

(Alternatively: **1** for differentiating and getting $t = 2$ and **1** for min. distance.)

C3.



(a) Resolving forces horizontally

$$T_1 \cos 45^\circ = T_2 \cos 30^\circ \qquad \mathbf{1}$$

$$\frac{T_1}{\sqrt{2}} = \frac{\sqrt{3}}{2} T_2$$

$$T_1 = \frac{\sqrt{3}}{\sqrt{2}} T_2 \qquad \mathbf{1}$$

(b) Resolving vertically

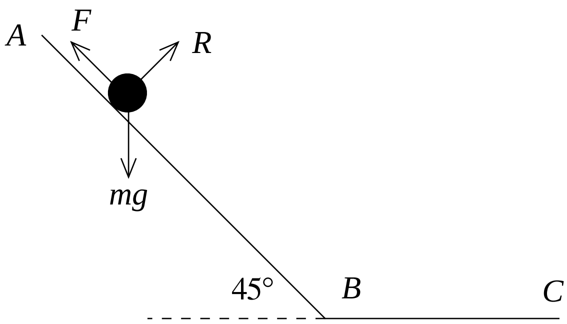
$$T_1 \sin 45^\circ = 1000g + T_2 \sin 30^\circ \qquad \mathbf{1}$$

$$\frac{1}{\sqrt{2}} T_1 - \frac{1}{2} T_2 = 1000g$$

$$\frac{1}{2}\left(\sqrt{3} - 1\right) T_2 = 1000g \qquad \mathbf{1}$$

$$T_2 = \frac{2000g}{\sqrt{3} - 1} \approx 26774 \text{ N} \qquad \mathbf{1}$$

C4.



(a) Resolving perpendicular to the chute gives $R = \frac{1}{\sqrt{2}} mg$ so
$F = \frac{1}{2} \times \frac{1}{\sqrt{2}} mg = \frac{mg}{2\sqrt{2}}$ **1**

Over section *AB*, applying Newton II

$$ma = mg \sin 45^\circ - \frac{1}{2\sqrt{2}} mg \qquad \mathbf{1}$$

$$\Rightarrow \quad a = \frac{g}{2\sqrt{2}}. \qquad \mathbf{1}$$

The speed of Jill at *B*, v_B, is given by $v_B^2 = 2aL = \frac{gL}{\sqrt{2}} \Rightarrow v_B = \sqrt{\frac{gL}{\sqrt{2}}}$. **1**

(b) Over the section *BC*, applying Newton II

$$ma_{BC} = -\tfrac{1}{2} mg$$

$$a_{BC} = -\frac{1}{2} g. \qquad \mathbf{1}$$

so that at *C*

$$v_C^2 = \frac{gL}{\sqrt{2}} + 2\left(\frac{-g}{2}\right) \times \frac{L}{2} \qquad \mathbf{1}$$

$$= \frac{gL}{2}\left(\sqrt{2} - 1\right) \qquad \mathbf{1}$$

$$\Rightarrow \quad v_C = \sqrt{\frac{gL}{2}\left(\sqrt{2} - 1\right)}.$$

C5. (a) For $0 \leqslant t < T$

$$mv = \int_0^t F\,dt \qquad \mathbf{1}$$

$$\Rightarrow v = \frac{Ft}{m}. \qquad \mathbf{1}$$

$$\text{For } t \geqslant T, \qquad V = \frac{FT}{m} \qquad \mathbf{1}$$

[Alternatively: $v = at \Rightarrow a = \frac{v}{t}$ and $F = ma \Rightarrow v = \frac{Ft}{m}$, as F is constant.]

(b)

$$W = \int_0^T F\,ds = \int_0^T F\frac{ds}{dt}\,dt = \int_0^T Fv\,dt \qquad \mathbf{1}$$

$$= \frac{F^2}{m}\int_0^T t\,dt \qquad \mathbf{1}$$

$$= \frac{F^2T^2}{2m} \qquad \mathbf{1}$$

C6. (a) By Newton II

$$m\frac{dv}{dt}\mathbf{i} = -0.05mv\mathbf{i} \qquad \mathbf{v}(0) = 2\mathbf{i}$$

$$\Rightarrow \quad \frac{dv}{dt} = -0.05v, \qquad v(0) = 2. \qquad \mathbf{1}$$

Separating the variables

$$\int\frac{dv}{v} = -0.05t + c \qquad \mathbf{1}$$

$$\Rightarrow \quad \ln|v| = -0.05t + c$$

$$\Rightarrow \quad v = e^{-0.05t + c} \qquad \mathbf{1}$$

Since $v(0) = 2$, $e^c = 2$ and hence **1**

$$v(t) = 2e^{-0.05t}.$$

(b) When $v = 1$, $e^{-0.05t} = \frac{1}{2}$. **1**

$$\Rightarrow t = 20\ln 2$$

$$= 13.9 \text{ to 1 decimal place} \qquad \mathbf{1}$$

C7. (a) $\mathbf{V} = V(\cos 30°\mathbf{i} + \sin 30°\mathbf{j}) = \frac{1}{2}V(\sqrt{3}\mathbf{i} + \mathbf{j})$ or for V_y only. **1**

The y-component of the equation of motion gives

$$\ddot{y} = -g \Rightarrow \dot{y} = \frac{V}{2} - gt$$ **1**

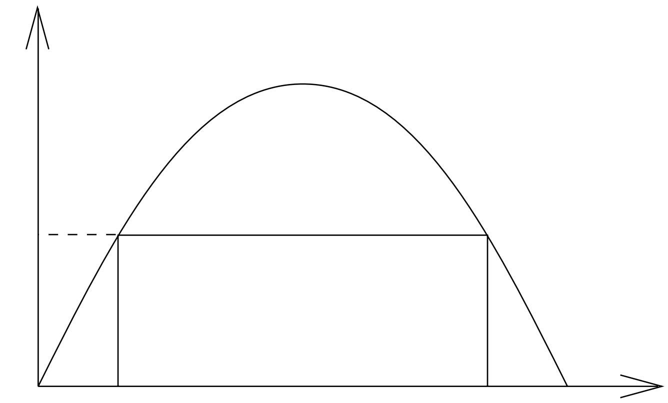
$$\Rightarrow y = \frac{Vt}{2} - \frac{1}{2}gt^2 = \frac{t}{2}(V - gt).$$ **1**

(b) Note that $\dot{y} = \frac{1}{2}V - gt$ so the maximum height occurs when $t = \frac{V}{2g}$. **1**

Hence

$$y_{\max} = \frac{V}{4g}\left(V - \frac{V}{2}\right) = \frac{V^2}{8g}.$$ **1**

(c)



We need the times when $y = \frac{1}{4}y_{\max}$.

$$\Rightarrow \frac{1}{2}Vt - \frac{1}{2}gt^2 = \frac{V^2}{32g}$$ **1**

$$\Rightarrow t^2 - \frac{V}{g}t + \frac{V^2}{16g^2} = 0$$ **1**

$$\Rightarrow t = \frac{1}{2}\left[\frac{V}{g} \pm \left(\frac{V^2}{g^2} - \frac{V^2}{4g^2}\right)^{1/2}\right]$$ **1**

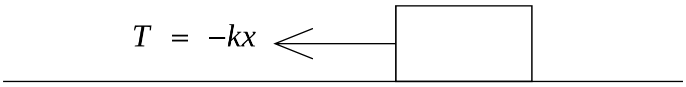
$$= \frac{V}{2g}\left[1 \pm \frac{\sqrt{3}}{2}\right]$$ **1**

The time the missile appears on the radar is

$$\frac{V}{2g}\left[1 + \frac{\sqrt{3}}{2}\right] - \frac{V}{2g}\left[1 - \frac{\sqrt{3}}{2}\right]$$ **1**

$$= \frac{\sqrt{3}V}{2g}.$$

C8. (a)



By Newton II

$$m\frac{d^2x}{dt^2} = -kx \qquad \mathbf{1}$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = \frac{-k}{m}x$$

$$= -\omega^2 x, \text{ where } \omega^2 = \frac{k}{m} \qquad \mathbf{1}$$

$$\left(x(0) = a,\ \dot{x}(0) = 0\right)$$

Noting that $\frac{d^2x}{dt^2} = v\frac{dv}{dx}$ then

$$v\frac{dv}{dx} = -\omega^2 x \qquad \mathbf{1}$$

$$\Rightarrow \quad \int v\,dv = -\omega^2 \int x\,dx$$

$$\Rightarrow \quad \frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + c \qquad \mathbf{1}$$

Since $v = 0$ when $x = a$, then $2c = \omega^2 a^2$, so **1**

$$v^2 = \omega^2\left(a^2 - x^2\right).$$

(b) The P.E. in the spring is $E_p = \frac{1}{2}kx^2$
so we have

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}kx^2 \qquad \mathbf{1}$$

$$v^2 = \frac{k}{m}x^2$$

$$\Rightarrow \omega^2\left(a^2 - x^2\right) = \omega^2 x^2 \qquad \mathbf{1}$$

$$\Rightarrow \quad x^2 = \frac{a^2}{2}$$

$$\Rightarrow \quad x = \pm\frac{a}{\sqrt{2}} \qquad \mathbf{1}$$

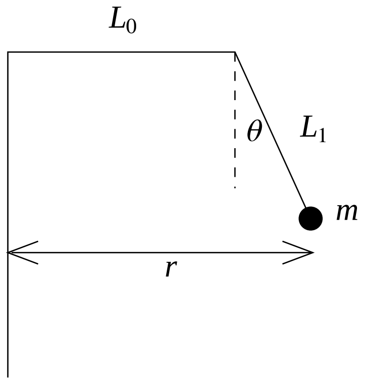
Now $x = a\cos\omega t$ thus $\cos\omega t = \pm\frac{1}{\sqrt{2}}$ **1**

so $\cos\omega t = \frac{1}{\sqrt{2}} \Rightarrow \omega t = \frac{\pi}{4}$

and $\cos\omega t = -\frac{1}{\sqrt{2}} \Rightarrow \omega t = \frac{3\pi}{4}$ **1**

Time taken to travel between $x = \pm\frac{a}{\sqrt{2}}$ is $\frac{\pi}{2\omega}$. **1**

C9.



(a) Note that $r = L_0 + L_1 \sin\theta$ (*). **1**

Let T be the tension in the chain.

Resolving vertically

$$T \cos\theta = mg$$

$$T = \frac{mg}{\cos\theta} \qquad (**)$$ **1**

Resolving horizontally and using Newton II

$$ma = T \sin\theta$$

$$\Rightarrow \quad mr\omega^2 = T \sin\theta$$ **1**

$$\Rightarrow \quad \omega^2 = \frac{mg}{\cos\theta} \times \frac{\sin\theta}{mr}$$ **1**

$$= \frac{g \tan\theta}{r}$$

and from (*)

$$\omega^2 = \frac{g \tan\theta}{L_0 + L_1 \sin\theta}$$ **1**

(b) When $\theta = 30°$

$$\omega_1^2 = \frac{\frac{1}{\sqrt{3}} g}{L_0 + 2L_0 \times \frac{1}{2}}$$ **1**

$$= \frac{g}{L_0} \times \frac{1}{2\sqrt{3}}$$ **1**

When $\theta = 60°$

$$\omega_2^2 = \frac{\sqrt{3}\, g}{L_0 + 2L_0 \times \frac{\sqrt{3}}{2}}$$

$$= \frac{g}{L_0} \times \frac{\sqrt{3}}{1 + \sqrt{3}}$$ **11**

Dividing these equations:

$$\frac{\omega_2^2}{\omega_1^2} = \frac{\sqrt{3}}{1 + \sqrt{3}} \times 2\sqrt{3} = \frac{6}{1 + \sqrt{3}}$$ **1**

$$\Rightarrow \omega_2^2 = \frac{6}{1 + \sqrt{3}} \omega_1^2$$

[END OF MARKING INSTRUCTIONS]

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2003 Applied Mathematics

Advanced Higher – Section D

Finalised Marking Instructions

Advanced Higher Applied 2003: Section D Solutions and marks

D1.

$$y = \frac{\cos x}{1 - \sin x}$$

$$\frac{dy}{dx} = \frac{-\sin x(1 - \sin x) - \cos x(-\cos x)}{(1 - \sin x)^2}$$ **1M,1**

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2}$$

$$= \frac{1 - \sin x}{(1 - \sin x)^2} = \frac{1}{1 - \sin x}.$$ **1**

D2.

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 2 & -1 & 1 & 0 \\ 1 & 1 & -\frac{1}{2} & 1 \end{array}\right) \quad \left(\begin{array}{ccc|cl} 1 & 1 & -1 & 3 & \\ 0 & -3 & 3 & -6 & r_2 - 2r_1 \\ 0 & 0 & \frac{1}{2} & -2 & r_3 - r_1 \end{array}\right)$$ **2E1**

$$z = -4$$ **1**

$$-3y - 12 = -6 \Rightarrow y = -2$$

$$x - 2 + 4 = 3 \Rightarrow x = 1$$ **1**

D3.

$$\frac{3x^2 + 2}{(x + 2)^2} = \frac{3x^2 + 2}{x^2 + 4x + 4}$$

$$\begin{array}{r|lll} & & & 3 \\ \hline x^2 + 4x + 4 & 3x^2 & & +2 \\ & 3x^2 & +12x & +12 \\ \hline & & -12x & -10 \end{array}$$ **1M**

So $$\frac{3x^2 + 2}{(x + 2)^2} = 3 - \frac{12x + 10}{(x + 2)^2}$$ **1**

Now write $\frac{12x + 10}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$.

$$12x + 10 = A(x + 2) + B.$$

Equating coefficients $A = 12$ **M1**

$$2A + B = 10 \Rightarrow B = -14$$ **1**

$$\frac{3x^2 + 2}{(x + 2)^2} = 3 - \frac{12}{(x + 2)} + \frac{14}{(x + 2)^2}$$ **1**

D4.

$$(3x - 2y)^4 = (3x)^4 + 4(3x)^3(-2y) + 6(3x)^2(-2y)^2 + 4(3x)(-2y)^3 + (-2y)^4$$ **2E1**

$$= 81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4$$ **1**

When $y = \frac{1}{x}$, the term which is independent of x is 216. **1**

D5. (a)

$$\int \frac{2e^x}{1 + e^x}\,dx = 2\int \frac{e^x}{1 + e^x}\,dx \qquad \mathbf{1}$$

$$= 2\ln(1 + e^x) + c. \qquad \mathbf{1}$$

(b)

$$u = 1 - \sin x \Rightarrow du = -\cos x\,dx \qquad \mathbf{1}$$

and when $x = 0,\ u = 1;\ x = \frac{\pi}{6},\ u = 1 - \frac{1}{2} = \frac{1}{2}.$ **1**

$$\int_0^{\pi/6} \frac{\cos x}{(1 - \sin x)^{3/2}}\,dx = \int_1^{1/2} -u^{-3/2}\,du \qquad \mathbf{1}$$

$$= -\left[\frac{u^{-1/2}}{-\frac{1}{2}}\right]_1^{1/2} = \left[\frac{2}{\sqrt{u}}\right]_1^{1/2} \qquad \mathbf{1}$$

$$= \left[2\sqrt{2} - 2\right] \qquad \mathbf{1}$$

$$= 2(\sqrt{2} - 1) \approx 0.828.$$

D6. (a)

$$f(x) = \frac{x^3 - 8x^2 + 16x + 4}{x^2 - 8x + 16}$$

$$\begin{array}{r|l} & x \\ x^2 - 8x + 16 & x^3 \ -8x^2 \ +16x \ +4 \\ & x^3 \ -8x^2 \ +16x \\ \hline & \qquad\qquad\quad 4 \end{array} \qquad \mathbf{1}$$

So $f(x) = x + \dfrac{4}{(x - 4)^2}$ (i.e. $a = 1$ and $b = 4$.) **1**

The vertical asymptote is $x = 4$. **1**

The non-vertical asymptote is $y = x$. **1**

(b)

$$f(x) = x + 4(x - 4)^{-2}$$

$f'(x) = 1 - 8(x - 4)^{-3} = 0$ at stationary values **1**

$$\frac{8}{(x - 4)^3} = 1 \Rightarrow (x - 4)^3 = 8 \Rightarrow x = 6 \qquad \mathbf{1}$$

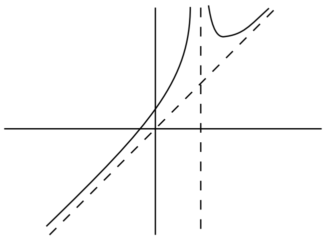
(or $x^2 - 6x + 12 = 0$, non-real)

i.e. Just one turning point.

$$f'(x) = 1 - \frac{8}{(x - 4)^3} \Rightarrow f''(x) = \frac{24}{(x - 4)^4} > 0 \qquad \mathbf{1}$$

The turning point is minimum at $(6,\ 7)$. **1**

(c)



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[END OF MARKING INSTRUCTIONS]

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2003 Applied Mathematics

Advanced Higher – Section E

Finalised Marking Instructions

E1. P(Taxi Yellow | Witness states Yellow)

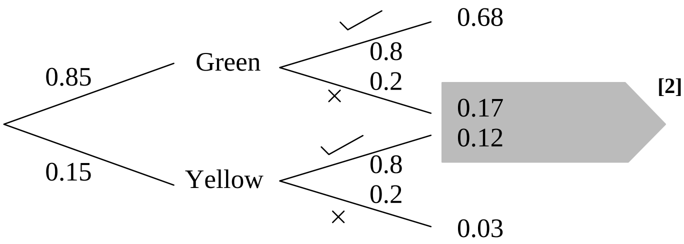
$= \dfrac{\text{P(Taxi Yellow} \cap \text{Witness states Yellow)}}{\text{P(Witness states Yellow)}}$ **M1**

$= \dfrac{\text{P(Witness states Yellow | Taxi Yellow).P(Taxi Yellow)}}{\text{P(WY | TY).P(TY) + P(WY | TG).P(TG)}}$ **M1**

$= \dfrac{0.8 \times 0.15}{0.8 \times 0.15 + 0.2 \times 0.85}$ **1, 1**

$= \dfrac{0.12}{0.29} = 0.41$ **1**

Alternative



$\dfrac{0.12}{0.17 + 0.12} = \dfrac{12}{29}$ [1] [1] [1]

E2. T_1 has mean 180 and s.d. 12
T_2 has mean 20 and s.d. 5

Total $T = T_1 + T_2$ has mean 180 + 20 = 200 **1**
variance $12^2 + 5^2 = 169$ **1**
so s.d. $= 13$ **1**

Assuming that T is normally distributed **1**
the required value of z is 3.09 **1**

$T = \mu + z\sigma = 200 + 3.09 \times 13 = 240.17$
$x = 241$ **1**

E3. (a) Assemble the lists into a single list of pupils numbered 1 to 180. **1**

Required sample size is 10% of 180 = 18.

Select a number at random from 1 to 10 e.g. 3 **1**

Take the 3rd pupil from the list and every 10th pupil thereafter. **1**

(b) Possible advantages: (just one needed) **1**

- systematic sampling is easily implemented
- helps to provide a good spread

Possible disadvantages: (just one needed) **1**

- if there is a pattern in the population matched by the sequence an unrepresentative sample will result
- in order to number the population, its size has to be known.

E4.

$X \sim Bin(100,\ 0.75)$ **1**

$\Rightarrow X$ is approximately N **1**

and $N(75,\ 4.33^2)$ **1**

$P(X \leqslant 70) = P\left(Z \leqslant \dfrac{70.5 - 75}{4.33}\right)$ **1**

$= P(Z \leqslant -1.04)$ **1**

$= 0.1492$ **1**

E5.

(a) Assume that the journey time is normally distributed (with $\sigma = 3$). **1**

$H_0 : \mu = 28$

$H_1 : \mu \neq 28$ [Must be two-tailed] **1**

$z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{25.125 - 28}{\frac{3}{\sqrt{8}}}$

$= -2.71$ **1**

The critical region is $z < -2.58$ or $z > 2.58$. **1**

Since $-2.71 < -2.58$ the null hypothesis would be rejected **1**

at the 1% level of significance i.e. there is evidence of a change. **1**

(b) p-value = $2 \times \Phi(-2.71)$ **1**

$= 2(1 - 0.9966) = 0.0068$ **1**

The fact that the p-value is less than 0.01 confirms rejection of the null hypothesis at the 1% level of significance **1**

(c) The fact that 28 does not lie in the 99% confidence interval confirms rejection of the null hypothesis at the 1% level. **1**

[END OF MARKING INSTRUCTIONS]

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2003 Applied Mathematics

Advanced Higher – Section F

Finalised Marking Instructions

Advanced Higher Applied 2003: Section F Solutions and marks

F1. $f(x) = \sqrt{9-4x}, \quad f'(x) = \dfrac{-2}{(9-4x)^{1/2}} \quad f''(x) = \dfrac{-4}{(9-4x)^{3/2}} \quad f'''(x) = \dfrac{-24}{(9-4x)^{5/2}}$

Taylor polynomial is

$$p(2+h) = 1 - 2h - \frac{4h^2}{2} - \frac{24h^3}{6}$$
$$= 1 - 2h - 2h^2 - 4h^3. \qquad \textbf{3}$$

Second degree approximation is $p(2 + 0{\cdot}03) = 1 - 0{\cdot}06 - 0{\cdot}0018 = 0{\cdot}9382$ **2**

Principal truncation error term is $-4 \times 0{\cdot}03^3 = -0{\cdot}0001$.
Hence second order estimate cannot be guaranteed accurate to 4 decimal places. **2**

F2. $$L(x) = \frac{(x-0{\cdot}2)(x-0{\cdot}5)}{(-0{\cdot}2)(-0{\cdot}5)}1{\cdot}306 + \frac{(x-0{\cdot}0)(x-0{\cdot}5)}{(0{\cdot}2)(-0{\cdot}3)}1{\cdot}102 + \frac{(x-0{\cdot}0)(x-0{\cdot}2)}{(0{\cdot}5)(0{\cdot}3)}0{\cdot}741$$
$$= (x^2 - 0{\cdot}7x + 0{\cdot}1)13{\cdot}06 - (x^2 - 0{\cdot}5x)18{\cdot}367 + (x^2 - 0{\cdot}2x)4{\cdot}490$$
$$= -0{\cdot}367x^2 - 0{\cdot}947x + 1{\cdot}306 \qquad \textbf{4}$$

F3. Let quadratic through $(x_0, f_0), (x_1, f_1), (x_2, f_2)$ be
$$y = A_0 + A_1(x - x_0) + A_2(x - x_0)(x - x_1).$$
Then $f_0 = A_0;\ f_1 = A_0 + A_1h;\ f_2 = A_0 + 2A_1h + 2A_2h^2$
and so
$$A_1 = \frac{f_1 - f_0}{h} = \frac{\Delta f_0}{h}; \qquad A_2 = \frac{f_2 - 2f_1 + f_0}{2h^2} = \frac{\Delta^2 f_0}{2h^2}.$$
Thus
$$y = f_0 + \frac{x - x_0}{h}\Delta f_0 + \frac{(x - x_0)(x - x_1)}{2h^2}\Delta^2 f_0.$$
Setting $x = x_0 + ph$ gives
$$y = f_0 + p\Delta f_0 + \tfrac{1}{2}p(p-1)\Delta^2 f_0. \qquad \textbf{5}$$
(Can also be done by an operator expansion of $(1 + \Delta)^p$.)

F4. (a) Maximum error is 8ε, i.e. $8 \times 0{\cdot}0005 = 0{\cdot}004$. **1**

(b) $\Delta^2 f_3 = 0{\cdot}167$. **1**

(c) Third degree polynomial would probably not be particularly good as an approximation as differences are not constant. **1**

(d) Working from $x = 2{\cdot}0$, $p = 0{\cdot}9$.

$$f(2{\cdot}18) = 2{\cdot}318 + 0{\cdot}9(0{\cdot}197) + \frac{(0{\cdot}9)(-0{\cdot}1)}{2}(0{\cdot}086)$$
$$= 2{\cdot}318 + 0{\cdot}177 - 0{\cdot}004 = 2{\cdot}491 \qquad \textbf{2}$$

F5. (a) Simpson's rule calculation is:

x	$f(x)$	m_1	$m_1 f(x)$	m_2	$m_2 f(x)$
0	0·0	1	0·0	1	0·0
0·25	0·04868			4	0·19472
0·5	0·15163	4	0·60653	2	0·30326
0·75	0·26571			4	1·06284
1	0·36788	1	0·36788	1	0·36788
			0·97441		1·92870

Hence $I_2 = 0{\cdot}97441 \times 0{\cdot}5/3 = 0{\cdot}16240$

and $I_4 = 1{\cdot}92870 \times 0{\cdot}25/3 = 0{\cdot}16072$ **4**

(b) $f^{\text{iv}}(0) = 12$; $f^{\text{iv}}(1) = 1{\cdot}84$.

Maximum truncation error $\approx 12 \times 0{\cdot}25^4/180 = 0{\cdot}00026$. **2**

Hence suitable estimate is $I_4 = 0{\cdot}161$. **1**

(c) With n strips and step size $2h$, the Taylor series for expansion of an integral approximated by Simpson's rule (with principal truncation error of $O(h^4)$) is

$I = I_n + C(2h)^4 + D(2h)^6 + \ldots = I_n + 16Ch^4 + \ldots$ (1)

With $2n$ strips and step size h, we have

$I = I_{2n} + Ch^4 + Dh^6 + \ldots$ (2)

$16 \times (2) - (1)$ gives $15I = 16I_{2n} - I_n + O(h^6)$

i.e. $I \approx (16I_{2n} - I_n)/15 = I_{2n} + (I_{2n} - I_n)/15$ **3**

$I = 0{\cdot}16072 + (0{\cdot}16072 - 0{\cdot}16240)/15 = 0{\cdot}16061$

or 0·1606 to suitable accuracy. **1**

[END OF MARKING INSTRUCTIONS]

2003 Applied Mathematics

Advanced Higher – Section G

Finalised Marking Instructions

Advanced Higher Applied 2003: Section G Solutions and marks

G1.

We are given that $\frac{d^2x}{dt^2} = 12 - 3t^2, v(0) = 0, s(0) = 0$

$$\Rightarrow \quad v(t) = 12t - t^3 \qquad \mathbf{1}$$

$$\Rightarrow \quad s(t) = 6t^2 - \frac{1}{4}t^4. \qquad \mathbf{1}$$

When the particle comes to rest

$$v(t) = 0 \quad \Rightarrow \quad 12t - t^3 = 0$$
$$\Rightarrow \quad t^2 = 0 \text{ or } t^2 = 12$$
$$\Rightarrow \quad t = 2\sqrt{3} \text{ (since } t > 0). \qquad \mathbf{1}$$

The position at this time is

$$s(2\sqrt{3}) = 6 \times 12 - \frac{1}{4} \times 12^2 = 72 - 36 = 36 \text{ m} \qquad \mathbf{1}$$

G2.

(a) Given $\mathbf{a}_A = -2\mathbf{j}$; $\mathbf{v}_A(0) = \mathbf{i}$; $\mathbf{r}_A(0) = -\mathbf{i}$

$$\mathbf{v}_A(t) = -2t\mathbf{j} + \mathbf{c} = \mathbf{i} - 2t\mathbf{j} \qquad \mathbf{1}$$

$$\Rightarrow \quad \mathbf{r}_A(t) = t\mathbf{i} - t^2\mathbf{j} - \mathbf{i} = (t-1)\mathbf{i} - t^2\mathbf{j} \qquad \mathbf{1}$$

(b) (i)

$$_A\mathbf{r}_B = \mathbf{r}_A - \mathbf{r}_B = (2-t)\mathbf{i} - \mathbf{j} \qquad \mathbf{1}$$

(ii) The square of the distance between A and B is

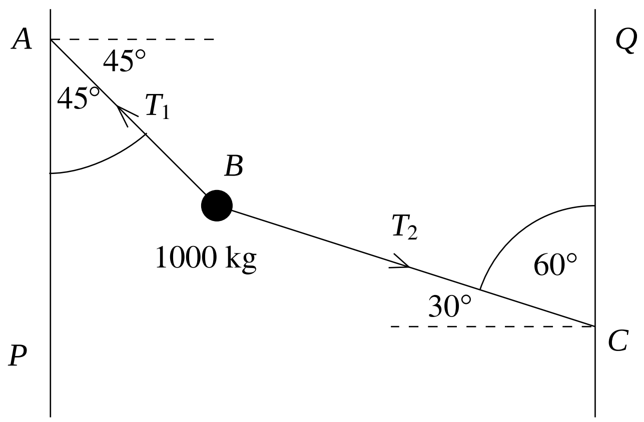
$$|_A\mathbf{r}_B|^2 = (2-t)^2 + 1. \qquad \mathbf{1}$$

This has minimum when $t = 2$, **1**

and the minimum distance is 1 metre. **1**

(Alternatively: **1** for differentiating and getting $t = 2$ and **1** for min. distance.)

G3.



(a) Resolving forces horizontally

$$T_1 \cos 45° = T_2 \cos 30° \qquad \mathbf{1}$$

$$\frac{T_1}{\sqrt{2}} = \frac{\sqrt{3}}{2} T_2$$

$$T_1 = \frac{\sqrt{3}}{\sqrt{2}} T_2 \qquad \mathbf{1}$$

(b) Resolving vertically

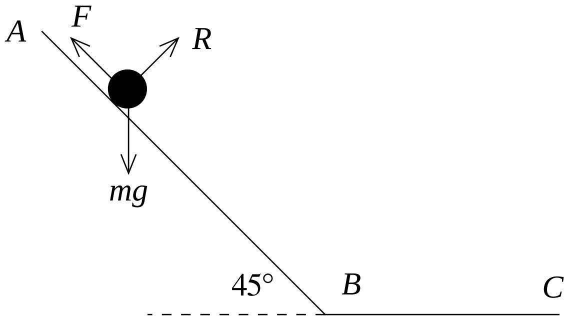
$$T_1 \sin 45° = 1000g + T_2 \sin 30° \quad \mathbf{1}$$

$$\frac{1}{\sqrt{2}}T_1 - \frac{1}{2}T_2 = 1000g$$

$$\frac{1}{2}\left(\sqrt{3} - 1\right)T_2 = 1000g \quad \mathbf{1}$$

$$T_2 = \frac{2000g}{\sqrt{3} - 1} \approx 26774 \text{ N} \quad \mathbf{1}$$

G4.



(a) Resolving perpendicular to the chute gives $R = \frac{1}{\sqrt{2}}mg$ so
$F = \frac{1}{2} \times \frac{1}{\sqrt{2}} mg = \frac{mg}{2\sqrt{2}}$ **1**

Over section AB, applying Newton II

$$ma = mg \sin 45° - \frac{1}{2\sqrt{2}}mg \quad \mathbf{1}$$

$$\Rightarrow \quad a = \frac{g}{2\sqrt{2}}. \quad \mathbf{1}$$

The speed of Jill at B, v_B, is given by $v_B^2 = 2aL = \frac{gL}{\sqrt{2}} \Rightarrow v_B = \sqrt{\frac{gL}{\sqrt{2}}}$. **1**

(b) Over the section BC, applying Newton II

$$ma_{BC} = -\tfrac{1}{2}mg$$

$$a_{BC} = -\frac{1}{2}g. \quad \mathbf{1}$$

so that at C

$$v_C^2 = \frac{gL}{\sqrt{2}} + 2\left(\frac{-g}{2}\right) \times \frac{L}{2} \quad \mathbf{1}$$

$$= \frac{gL}{2}\left(\sqrt{2} - 1\right) \quad \mathbf{1}$$

$$\Rightarrow \quad v_C = \sqrt{\frac{gL}{2}\left(\sqrt{2} - 1\right)}.$$

G5. (a) $\mathbf{V} = V(\cos 30°\mathbf{i} + \sin 30°\mathbf{j}) = \frac{1}{2}V(\sqrt{3}\mathbf{i} + \mathbf{j})$ or for V_y only. **1**

The y-component of the equation of motion gives

$$\ddot{y} = -g \Rightarrow \dot{y} = \frac{V}{2} - gt \qquad \mathbf{1}$$

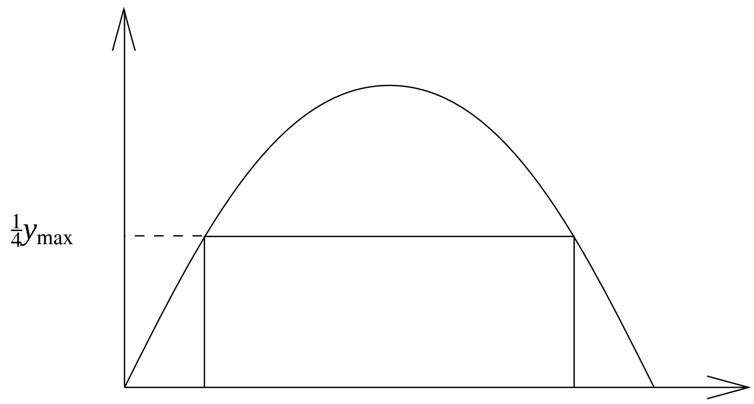
$$\Rightarrow y = \frac{Vt}{2} - \frac{1}{2}gt^2 = \frac{t}{2}(V - gt). \qquad \mathbf{1}$$

(b) Note that $\dot{y} = \frac{1}{2}V - gt$ so the maximum height occurs when $t = \frac{V}{2g}$. **1**

Hence

$$y_{\max} = \frac{V}{4g}\left(V - \frac{V}{2}\right) = \frac{V^2}{8g}. \qquad \mathbf{1}$$

(c)



We need the times when $y = \frac{1}{4}y_{\max}$.

$$\Rightarrow \frac{1}{2}Vt - \frac{1}{2}gt^2 = \frac{V^2}{32g} \qquad \mathbf{1}$$

$$\Rightarrow t^2 - \frac{V}{g}t + \frac{V^2}{16g^2} = 0 \qquad \mathbf{1}$$

$$\Rightarrow t = \frac{1}{2}\left[\frac{V}{g} \pm \left(\frac{V^2}{g^2} - \frac{V^2}{4g^2}\right)^{1/2}\right] \qquad \mathbf{1}$$

$$= \frac{V}{2g}\left[1 \pm \frac{\sqrt{3}}{2}\right] \qquad \mathbf{1}$$

The time the missile appears on the radar is

$$\frac{V}{2g}\left[1 + \frac{\sqrt{3}}{2}\right] - \frac{V}{2g}\left[1 - \frac{\sqrt{3}}{2}\right] \qquad \mathbf{1}$$

$$= \frac{\sqrt{3}V}{2g}.$$

[END OF MARKING INSTRUCTIONS]